DEPARTMENT OF MATHEMATICS

TECHNICAL REPORT

ALCIE SLOWLY DEVELOPS SELF-EFFICACY WITH WRITING PROOF FRAMEWORKS, BUT HER INITIAL PROGRESS AND SENSE OF SELF-EFFICACY EVAPORATES WHEN SHE ENCOUNTERS UNFAMILIAR CONCEPTS: HOWEVER, IT EVENTUALLY RETURNS[1]

AHMED BENKHALTI, DR. ANNIE SELDEN[2],
AND DR. JOHN SELDEN

SPRING 2017

No. 2017-1

TENNESSEE TECHNOLOGICAL UNIVERSITY
Cookeville, TN 38505

Alice Slowly Develops Self-Efficacy with Writing Proof Frameworks, but Her Initial Progress and Sense of Self-Efficacy Evaporates When She Encounters Unfamiliar Concepts: However, It Eventually Returns[1]

Ahmed Benkhalti
New Mexico State University

Annie Selden[2]
New Mexico State University

John Selden
New Mexico State University

We document Alice's progression with proof-writing over two semesters. We analyzed videotapes of her one-on-one sessions working through the course notes for our inquiry-based transition-to-proof course. Our theoretical perspective informed our work and includes the view that proof construction is a sequence of mental and physical, actions. It also includes the use of proof frameworks as a means of getting started. Alice's early reluctance to use proof frameworks, after an initial introduction to them, is documented, as well as her subsequent acceptance of and proficiency with them by the end of the real analysis section of the course notes, along with a sense of self-efficacy. However, during the second semester, upon first encountering semigroups, with which she had no prior experience, her proof writing deteriorated, as she coped with understanding the new concepts. But later, she began using proof frameworks again and regained a sense of self-efficacy.

Key words: Transition-to-proof, Proof Construction, Proof Frameworks, Self-efficacy, Coping with Abstraction, Working Memory

This case study focuses on how one non-traditional mature individual, Alice, in one-on-one sessions, progressed from an initial reluctance to use the technique of proof frameworks (Selden & Selden, 1995; Selden, Benkhalti, & Selden, 2014) to a gradual acceptance of, and eventual proficiency with, both writing proof frameworks and completing many entire proofs with familiar content. We also consider how this approach to proof construction helped this individual gain a sense of self-efficacy (Bandura, 1994, 1995) with regard to proving, but later evaporated upon encountering unfamiliar abstract concepts. However, after some time she was able to return to using the technique of proof frameworks and regained a sense of self-efficacy. This study also further illuminates the well-known, documented tendency of students to write proofs from the top-down, who consequently are often unable to develop complete proofs.

Theoretical Perspective

In our analysis and in our teaching, we consider proof construction to be a sequence of mental and physical actions, some of which do not appear in the final written proof text. Such a sequence of actions is related to, and extends, what has been called a "possible construction

[1] This paper is based on two presentation on Alice given at the 19th and 20th Conferences on Research in Mathematics Education in 2016 and 2017.

[2] Annie Selden is Professor Emerita of Mathematics from Tennessee Technological University. John Selden is retired from and also formerly taught mathematics at Tennessee Technological University. The Seldens have retired to Las Cruces, New Mexico and are now Adjunct Professors of Mathematics at New Mexico State University. Ahmed Benkhalti an interdisciplinary PhD student of theirs at New Mexico State University.

path" of a proof, illustrated in Selden and Selden (2009). For example, suppose that in a partly completed proof, there is an "or" in the hypothesis of a statement yet to be proved: "*If A or B, then C.*" Here, the situation is having to prove this statement. The interpretation is realizing the need to prove *C* by cases. The resulting action is constructing two independent sub-proofs; one in which one supposes *A* and proves *C*, the other in which one supposes *B* and proves *C*.

When several similar situations are followed by similar actions, an *automated link* may be learned between such situations and actions. Subsequently, a situation can be followed by an action, without the need for any conscious processing between the two (Selden, McKee, & Selden, 2010). When a situation occurs together with an action well beyond the simple forming of a link through associative leaning, it may be "overlearned" and the action will then occur automatically in the presence of the situation (Bargh, 2014). In our course, we have observed that, with sufficient practice, many proving actions can become the result of the enactment of linked, and sometimes automated, situation-action pairs. We have called these automated situation-action pairs *behavioral schemas* (Selden, McKee, & Selden, 2010; Selden & Selden, 2008). Linking proving actions to triggering situations and automating those actions can considerably reduce the burden on working memory, a very limited resource, and this tends to reduce errors (Baddeley, 2000).

Related Research and Concepts

While studies of students' proving have been made before, they have *not* often focused on students' use of proof frameworks. For example, Selden and Selden (1987) examined errors and misconceptions in undergraduate abstract algebra students' proof attempts. They found instances of assuming the conclusion, proving the converse, improper use of symbols, misuse of theorems, and trouble with quantifiers. Similarly, Hazzan and Leron (1996) in their study of students' misuse of Lagrange's Theorem found that students often used the converse of a theorem as if it were true or invoked the theorem where it did not apply. Selden, McKee, and Selden (2010) reported instances of students' tendencies to write proofs from the top down and their reluctance to unpack and use the conclusion to structure their proofs. (See the case of Willy, who focused too soon on the hypothesis in Selden, McKee, & Selden, 2010, pp. 209-211).

The Formal-Rhetorical and Problem-Centered Parts of Proofs

Previously, we (Selden & Selden, 2013) introduced the idea of the *formal-rhetorical* part of a proof as the part of a proof that comes from unpacking and using the logical structure of the statement of the theorem, associated definitions, and previously proved theorems. In general, this part *does not* depend on a deep understanding of, or intuition about, the concepts involved or on genuine problem solving in the sense of Schoenfeld (1985, p .74). However, the *problem-centered* part of a proof *does* depend on genuine mathematical problem solving, intuition, and a deeper understanding of the concepts involved. A major portion of the formal-rhetorical part of a proof can consist of a proof framework.

Proof Frameworks

An early version of the idea of proof frameworks was introduced by us (Selden & Selden, 1995):

> By a *proof framework* we mean a representation of the "top-level" logical structure of a proof, which does not depend on detailed knowledge of the relevant mathematical concepts, but which is rich enough to allow the reconstruction of the statement to be proved or one equivalent to it. A written representation of a proof framework might be a sequence of statements, interspersed with blank spaces, with the potential for being expanded into a proof by additional argument. (p. 129).

We went on (Selden & Selden, 1995) to connect the ability to unpack the logical structure of mathematical statements with the ability to construct proof frameworks and with proof validation. We also pointed out that mental skills were involved. The learning and mastering of such mental skills can involve much mental energy and considerable working memory. (p. 132). While we did not state this explicitly at the time, in the sample validation in the Appendix, we did note that sometimes checking a sufficiently complex part of a proof might overload working memory and potentially lead to error. (p. 146).

The First and Second Levels of a Proof Framework

Later, we further developed the idea of proof frameworks, including that there are often both first-level and second-level proof frameworks. A *proof framework* is determined by just the logical structure of the theorem statement and associated definitions. The most common form of a theorem is: "If P, then Q", where P is the hypothesis and Q is the conclusion. In order to construct a proof framework for it, one takes the hypothesis of the theorem, "P", and writes, "Suppose P" to begin the proof. One then skips to the bottom of the page and writes "Therefore Q", leaving enough space for the rest of the proof to emerge in between. This produces the *first level* of a proof framework. At this point, a prover should focus on the conclusion and "unpack" its meaning. It may happen that the unpacked meaning of Q has the same logical form as the original theorem, that is, a statement with a hypothesis and a conclusion. In that case, one can repeat the above process, providing a *second-level* proof framework in the space between the first and last lines of the emerging proof. (For some examples, see Selden, Benkhalti, & Selden, 2014).

Finally, we do not claim that mathematicians should write proofs using this technique, but only that doing so will be helpful for novice students and that their mathematics professors will accept the results. As long as the logical flow and clarity of a proof submission is correct, it does not matter (and is impossible to recover) in which order the sentences were written. We now return to the problem of overloaded working memory that can occur when a proof construction, or a proof validation, is sufficiently complex.

Working Memory

It has been said that the "two major components of our cognitive architecture that are critical to [thinking and] learning are long-term memory and working memory" (Kalyuga, 2014). *Working memory* makes cognition possible but has a limited capacity that varies across individuals. It is associated with the conscious processing of information within one's focus of attention. However, working memory can only deal with several units, or chunks, of information at a time, especially when working with novel information (Cowan, 2001; Miller,1956). In contrast, *long-term memory* can be thought of as a learner-organized knowledge base that has essentially unlimited capacity and can be used to help alleviate the limited capacity of working memory (Ericsson & Kintsch, 1995). However, when working memory capacity is overloaded, errors and oversights are likely to occur.

Coping with Mathematical Abstraction and Formality

While the mathematics education research literature does not seem to have considered working memory overload during learning per se, there are a few studies of coping with abstractions. These could be reinterpreted as related to working memory overload causing confusion. For example, Hazzan (1999) investigated how Israeli freshman computer science students, taking their first course in abstract algebra in a "theorem-proof format", coped by "reducing the level of abstraction". Specifically, she found that they tended "to work on a lower level of abstraction than the one in which the concepts are introduced in class" (p. 75). For example, when doing homework on abstract groups, a student might actually be thinking of a familiar group like the integers under addition.

Further, Leron and Hazzan (1997) pointed out that students in mathematical problem-solving situations "often experience confusion and loss of meaning." (p. 265), and that students attempt to make sense of a problem situation "in order to better *cope* with it." (p. 267). While this coping perspective occurs at all levels, they stated that "the phenomena of confusion and loss of meaning are even more pronounced in college mathematics courses." (p. 282). They also suggested that more work on the coping perspective in mathematics education is needed. Indeed, somewhat similarly, Pinto and Tall (1999) considered two different university students' coping mechanisms when confronted with formal definitions and proofs in real analysis. These were the ideas of *giving meaning* to definitions using concept images versus *extracting meaning* from the formal definition via deduction. However, not many university level mathematics education studies have specifically considered students' coping perspectives.

Methodology: Conduct of the Study

We met regularly for individual 75-minute sessions with a mature working professional, Alice, who wanted to learn how to construct proofs. Alice followed the same course notes previously written for our inquiry-based transition-to-proof course[3] used with beginning mathematics graduate students who wanted extra practice in writing proofs. The sessions were almost entirely devoted to having Alice attempt to construct proofs in front of us, often thinking aloud, and to giving her feedback and advice on her work. The notes had been designed to provide graduate students with as many different kinds of proving experiences as possible and included practice writing the kinds of proofs often found in typical proof-based courses, such as some abstract algebra and some real analysis. The notes included theorems on some sets, functions, real analysis, and algebra, in that order.

Alice had a good undergraduate background in mathematics from some time ago and also had prior teaching experience. She only worked on proofs during the actual times we met. While she usually came to see us twice a week to work on constructing proofs. Sometimes, when her paid work got a bit overwhelming, she would take a week off. Thus, unlike the graduate students

[3] A description of the course and course notes can be found in Selden, McKee, and Selden (2010, p. 207).

who took the course as a one-semester 3-credit class, Alice worked with us on our course notes for two semesters at her own pace and did not want credit.

We met in a small seminar room with blackboards on three sides, and Alice constructed original proofs at the blackboard, eventually using the middle blackboard almost exclusively for her evolving proofs. After several meetings, she began to use the left board for definitions and the right board for scratch work. She did not seem shy or overly concerned with working at the board in front of us, and from the start, we developed a very collegial working relationship. She seemed to enjoy our interactions as she worked through the course notes. Thus, we gained greater than normal insight into her mode of working. We videotaped every session and took field notes on what Alice wrote on the three boards, along with her interactions with us. For this study, we reviewed the first and second semester videos and field notes several times, looking for progression in Alice's approach to constructing proofs.

Alice's Progression Through the First Semester

Our First Meeting with Alice

We introduced Alice to the idea of proof frameworks and explained in detail how and why we use them. We also introduced her to the idea of unpacking the conclusion and mentioned that proofs are not written from the top down by mathematicians. With guidance, she was able to prove "If $A \subseteq B$, then $A \cap C \subseteq B \cap C$." In addition, she worked three exercises on writing proof frameworks--one on elementary number theory and two on set equality. Near the end of this meeting, Alice produced a proof framework for the next theorem in the notes. We felt that she not only understood our rationale for using proof frameworks, but also how to construct them.

Our Second Meeting with Alice—Her Reluctance to Use Proof Frameworks Surfaces

At the beginning of the second meeting, Alice went to the middle board and produced the same proof framework that she written five days earlier at our first meeting (Figure 1).

Theorem: Let A, B, and C be sets. If $A \subseteq B$, then $C - B \subseteq C - A$. Proof: Let A, B, and C be sets. Suppose $A \subseteq B$. Suppose $x \in C - B$. So $x \in C$ and x is not an element of B.	Thus $x \in C$ and x is not in A. Therefore x is in $C - A$. Therefore, $C - B \subseteq C - A$.

Figure 1. A proof framework that Alice produced on the middle blackboard.

Then Alice stopped and after a long silence of 65 seconds, much to our surprise, said, "I have a question for you. I find it very difficult to see the framework. Let me show you how I do it, because somehow I get confused with the framework." We asked her what it was about the framework that was confusing, but she seemingly could not put it into words. So we encouraged her to write a proof the way she preferred. Thus, on the left board, Alice began to write the proof in her own way in top down fashion (Figure 2).

Theorem: Let A, B, and C be sets. If $A \subseteq B$, then $C - B \subseteq C - A$. Proof. Let A, B, C be sets. Suppose A is a subset of B. We need to prove that $C - B$ is a subset of $C - A$. Suppose $x \in C - B$. We need to prove that $x \in C - A$.

Figure 2. Alice's attempt at constructing a proof in her own way.

Then she then paused for 15 seconds, and said, "We need to have one more," and wrote into her proof attempt, **"and $x \in A$"** immediately below **"$x \in C - B$"**, indicating with a caret that **"and $x \in A$"** was also part of her supposition (Figure 3). Then, after a 35-second pause, she added to her proof attempt, "**Since $x \in A$ and A is a subset B. Then $x \in B$.**" Shortly thereafter, Alice quietly said, "Oh, a contradiction". This was followed by, "Yeah, 'cause x doesn't belong to B. Yeah, problem here." Then, after a ten second pause, Alice said, "The problem is right here, isn't it?" pointing and underlining "***B***" and the statement "**and $x \in A$**." We asked, "And what do you think that problem is?" Alice replied, "I assumed that [pointing to **"and $x \in A$"**], but I do not know. I only know this [pointing to **"A is a subset of B"**]. We replied, "So that's a good point you've made."

Theorem: Let A, B, and C be sets. If $A \subseteq B$, then $C - B \subseteq C - A$. Proof. Let A, B, C be sets. Suppose A is a subset of $\underline{\boldsymbol{B}}$. We need to prove that $C - B$ is a subset of $C - A$. Suppose $\boldsymbol{x \in C - B}$. We need to prove that $x \in C - A$. **<u>and $x \in A$</u>.**

Figure 3. Alice's adjustments to her proof attempt, done in her own way.

After that, for a few minutes, we talked about the structure of proofs, and why we use proof frameworks. Then we asked Alice to elaborate on why **"and $x \in A$"** is a problem. She said, "I didn't write it right. I should have said here [pointing to the blank space to the left of **"and $x \in A$"**] I'm going to make an assumption like **'Suppose x belongs to the A'**, and then since x belongs to the A and I know that A is a subset of B, **then the x will belong to the B**." She continued, "I also know that x belongs in the $C - B$, because I said it earlier. Then x belongs to the C but **x does not belong to the B**." To which one of us replied, "And then you said something. I thought I heard you say the word **'contradiction'**." Alice explained, "Yeah, I got a contradiction because then I'm saying here [pointing to the board] the x belongs to the B, and the x doesn't belong to the B." We agreed, and she offered, "That assumption [pointing to **"and $x \in A$"**] was bad." We then reiterated why proof frameworks are structured the way they are, and suggested that we could take Alice's original framework (Figure 2) and what Alice had written on the left board (Figure 3), and change the order to write a proof. We proceeded to help Alice do this. For the rest of the semester, Alice seemed more inclined to attempt to use proof frameworks.

Alice's Way of Working

By midway through the first semester, Alice had developed her own pattern of working. She would:

1. First write the statement of the theorem to be proved on the middle board.

2. Then look up in the course notes the definitions of terms that occurred in the theorem statement and write them exactly as stated on the left board.
3. Next underline the relevant portions of definitions to assist with writing the proof framework. However, we did not teach her to use this strategy. (See Figure 4).
4. Use the right board for scratch work as needed.

Alice continued this pattern of working into the second semester.

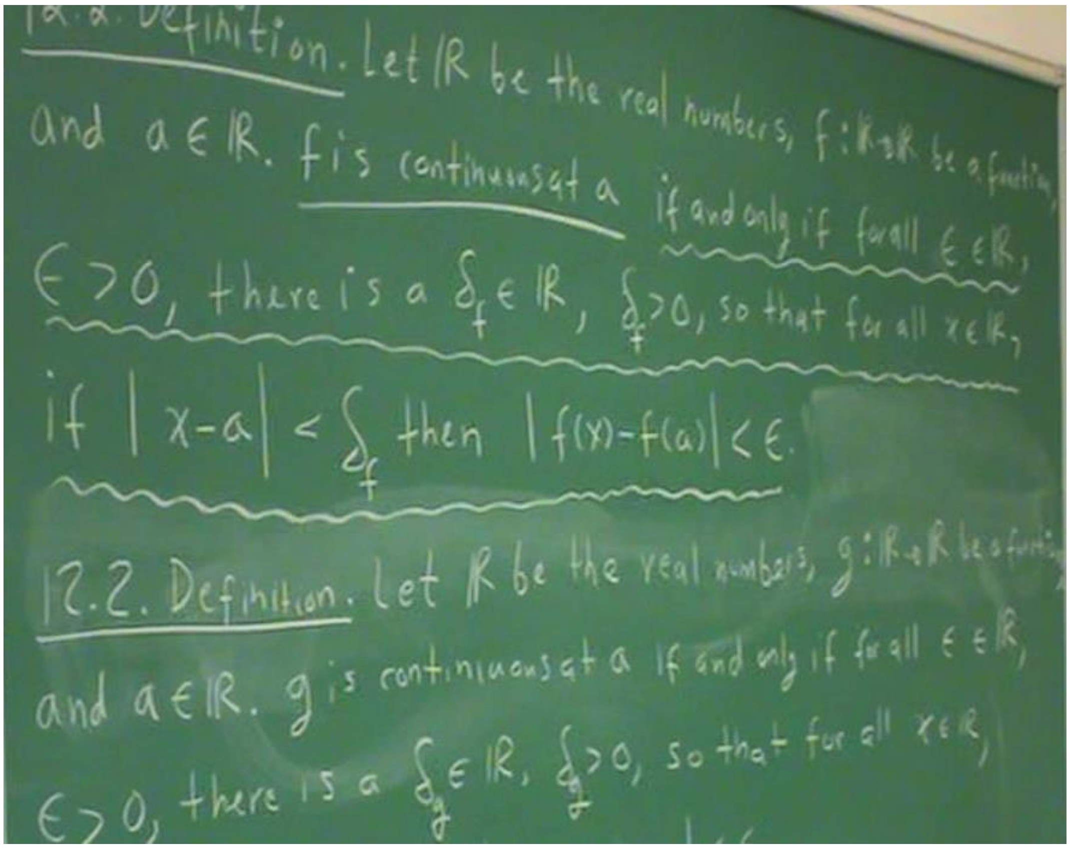

Figure 4. Alice's underlining of relevant parts of a definition.

Subsequent Meetings with Alice

As the first semester meetings went on, we observed that Alice became very methodical in her approach to proving, and also somewhat more accustomed to writing proof frameworks. We hypothesize this was because of her technical work experience and perhaps because of her natural tendencies. By the 12th meeting, Alice had developed the following pattern of working: She would write the statement of the theorem to be proved on the middle board, then look up in the course notes the definitions of terms that occurred in the theorem statement, write them exactly as stated on the left board, and use the right board for scratch work. Indeed, during the 12th meeting, when she got to the theorem, "*Let X, Y, and Z be sets. Let f: X→Y and g: Y→Z be 1-1 functions. Then g*○*f is 1-1*," she wrote the first- and second-level frameworks ostensibly on her own, and with some guidance from us, completed the proof and read it over for herself aloud.

By the 19th meeting at the end of the first semester, Alice was more fluent with writing proof frameworks than on the 12th meeting, and she had adopted the technique of writing definitions on the left board and changing the variable names to agree with those used in the theorem statement – all without prompting from us. This is remarkable as our experience has been that many students do not change variable names in definitions even when we suggest doing so, and this can often lead to difficulties. At this 19th meeting, Alice proved that the sum of two continuous functions is continuous (Figure 5). This proof has a rather complicated proof framework that necessitates leaving three blanks spaces -- one for using the hypothesis appropriately, one for specifying a δ, and one for showing that the chosen δ "works" (by showing the relevant distance is less than ε.)

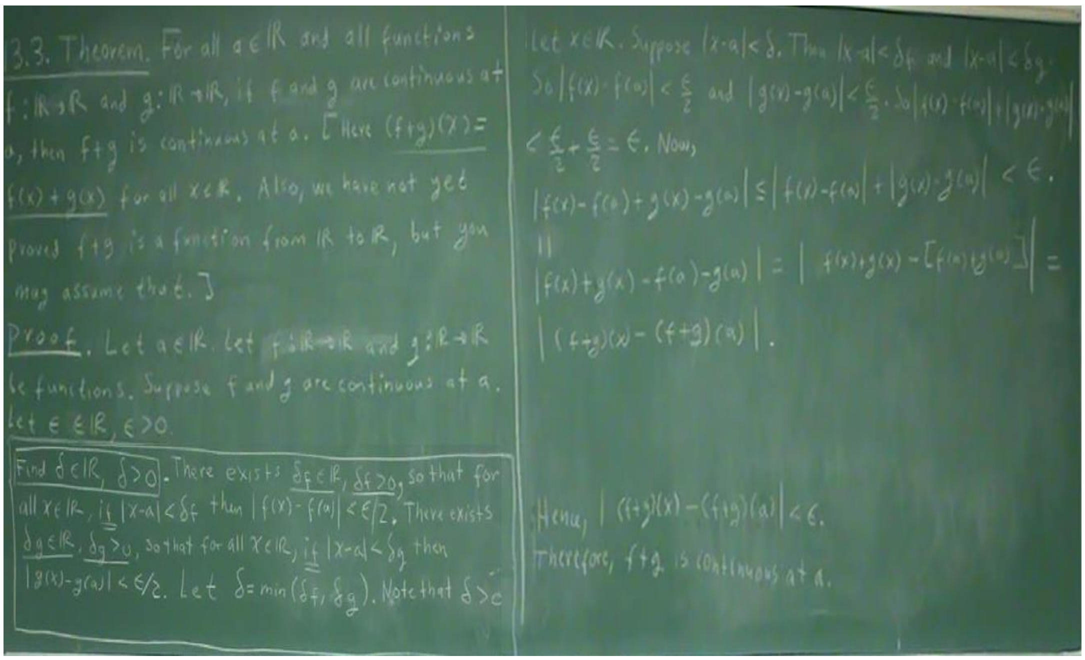

Figure 5. Alice's proof of that the sum of two continuous function is continuous.

Alice continued meeting with us and working on the course notes at her own pace during the second semester.

Alice's Progression through the Second Semester

Alice Continues Proving Real Analysis Theorems

Upon resuming in the second semester, Alice continued proving real analysis theorems, attempting to prove that the product of two continuous functions, f and g, is continuous in our first three meetings (i.e., our 20th-22nd meetings). She set up the proof framework correctly and explored the situation in scratch work. During this proving process, Alice made some astute observations, for example, having gotten to $|fg(x) - fg(a)| = |f(x)g(x) - f(a)g(a)| \leq |f(x)|\ |g(x) - g(a)| + |g(a)|\ |f(x) - f(a)|$, and having dealt with term involving $|g(a)|$, she noted that the former term was the "hard part" because $|f(x)|$, unlike $|g(a)|$, is not a constant. Somewhat later, Alice exhibited some self-monitoring, noting that she needed to move her sentence about the bound on $|f(x)|$, prior to setting δ equal to the "minimum of [the] three" deltas she had found. (See Figure 6). She also noted, in the 22nd meeting, that it seemed "weird" to write the restrictions on $|f(x) – f(a)|$ and $|g(x) – g(a)|$ without immediately explaining why she had chosen the bounds $\frac{\varepsilon}{2|g(a)|}$ and $\frac{\varepsilon}{2M_f}$, respectively, when applying the definition of continuity at *a* point to *f* and *g*.

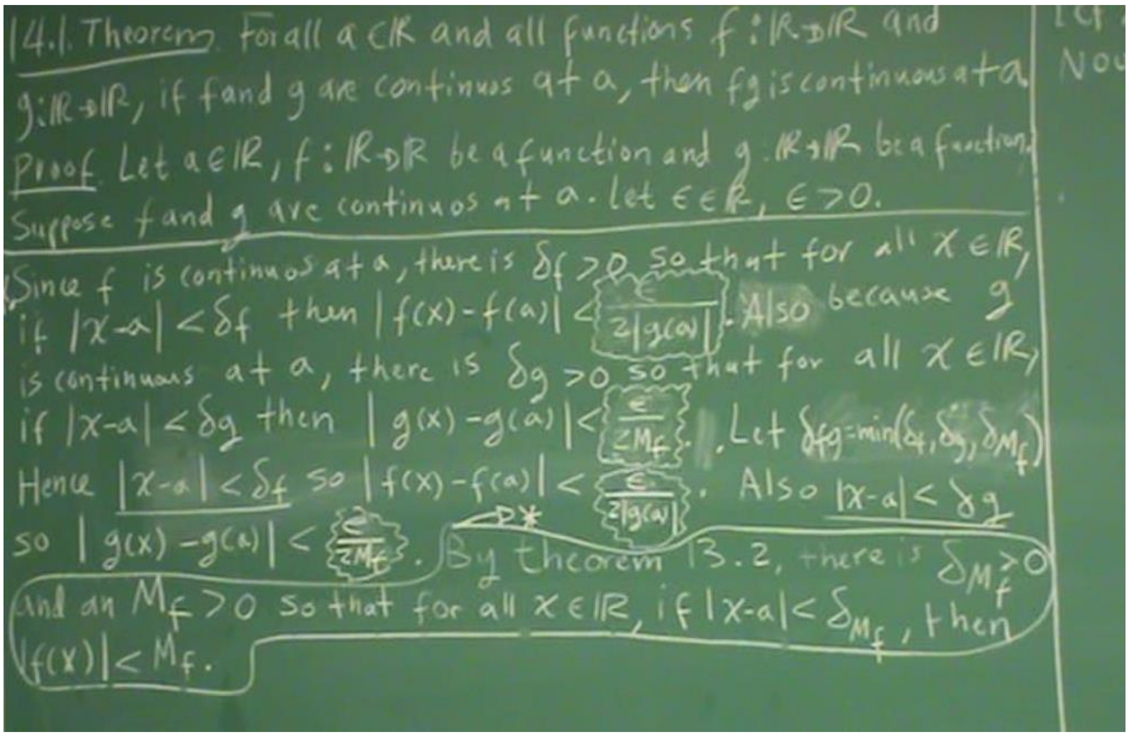

Figure 6. The first half of Alice's proof that the product of continuous functions is continuous, showing the "minimum of the three deltas".

Figure 7 shows the remainder of Alice's proof that the product of continuous functions is continuous, that is, that the her chosen δ "works".

Alice's Encounter with Semigroups Begins

By our 25th meeting, Alice had completed the real analysis section of the notes, and was ready to begin the abstract algebra (semigroups) section that starts with the definitions of binary operation and semigroup, followed by requests for examples. She provided only the most obvious of examples, such as the integers under addition or multiplication, and when asked for something "stranger", she said she could use the real numbers. When asked for another "strange" example "with no numbers at all", she suggested union as the binary operation, and with help, wrote up the example of the power set of a set of three elements. Next, when it came to providing examples of semigroups, she suggested the natural numbers with subtraction, but had to be prodded to check associativity; for this she considered (3 - 7) - 2 versus 3 - (7 - 2) and correctly inferred this was not a semigroup. To provide examples of left and right ideals, Alice needed to come up with a noncommutative semigroup, but she drew a blank. We suggested the semigroup of 2×2 real matrices under multiplication, and for an ideal, the subset of matrices of the form $\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix}$. After some calculation, Alice correctly concluded the subset is a right ideal, but not a left ideal.

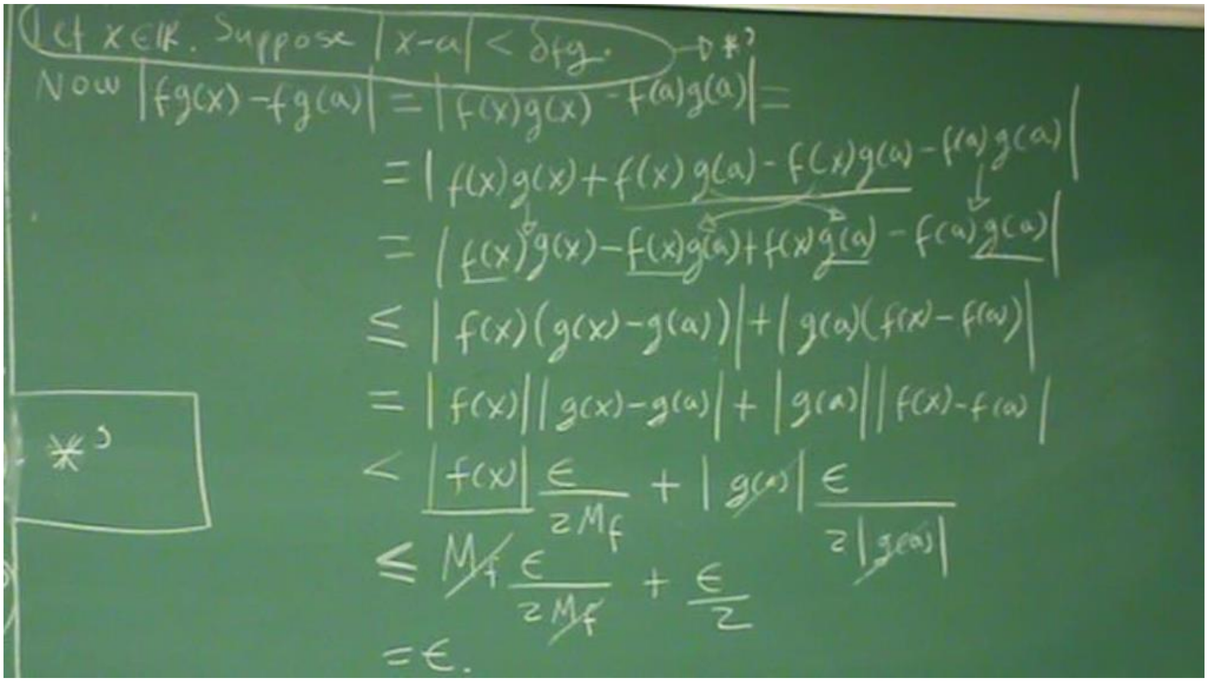

Figure 7. The remainder of Alice's proof that the product of continuous functions is continuous, showing that her chosen δ "works".

Alice Hits a "Brick Wall" with the First Semigroup Theorem

Alice continued considering examples for the first 35 minutes into the next (26th) meeting, after which she came to the first semigroup theorem to prove: "*Let S be a semigroup. Let L be a left ideal of S and R be a right ideal of S. Then* $L \cap R \neq \emptyset$." She first wrote the definitions of semigroup, left ideal, right ideal, and ideal on the left-hand board, as she had done many times before. Then she wrote the first-level framework on the middle board, after which she went to the right-hand board and began doing some scratch work, which included drawing a Venn diagram of two overlapping circles, *L* and *R*, with an arrow pointing to the intersection. She wrote in her scratch work "$L \cap R = \emptyset$" and "there exists an element $a \in L \cap R$". With this, it seemed that Alice was trying to clarify the theorem statement for herself. However, she had not yet attended to the second-level framework. We pointed this out.

During the rest of her proving attempt, we seemed to need to remind Alice of relevant actions, such as considering what she knew about ideals (i.e., that they are nonempty), and hence, concluding that each of *L* and *R* contains an element, which she labeled *l* and *r* respectively. Then, using those, she tried to "explore" to find an element in $L \cap R$, in order to conclude it was not empty. With our guidance, Alice finished the proof, but her sense of self-efficacy seemed shaken.

Indeed, at the next (27th) meeting, Alice wanted to reprove the theorem about the intersection of left and right ideals before continuing. (See Figure 8).We now feel that she had been somewhat overwhelmed, or confused, by the new content, perhaps causing working memory overload. She had tried to cope as best she could by concentrating on the new concepts, while "forgetting" her prior proof writing skills. Alice's hesitant behavior continued for eight more meetings.

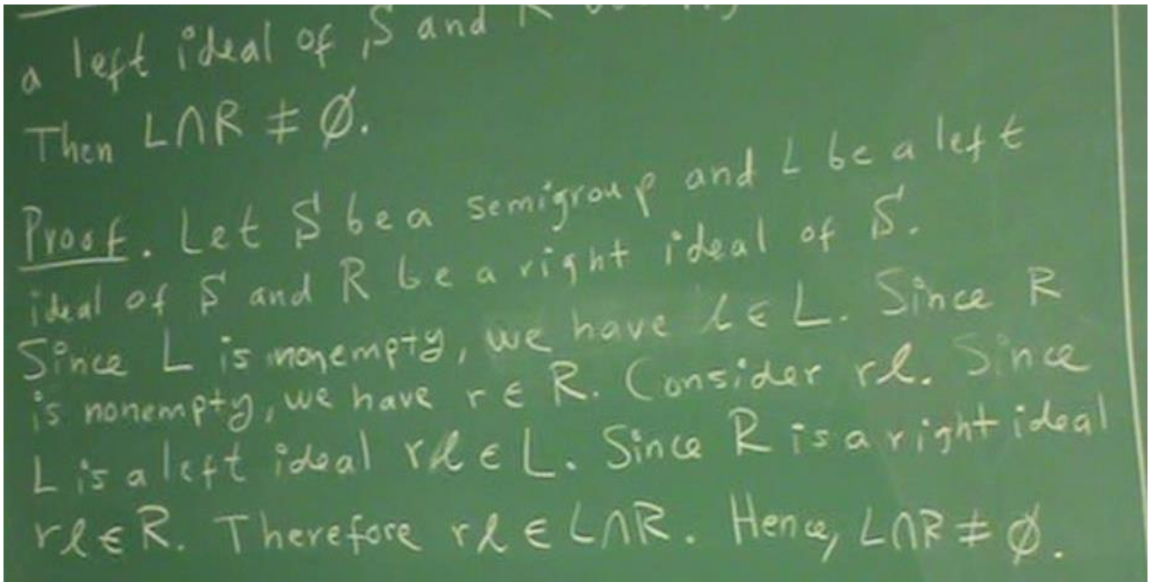

Figure 8. Alice's proof that the intersection of a left ideal and a right ideal is nonempty.

Alice Regains Her "Footing"

Then, during the 35th meeting, Alice considered the theorem, "*If S and T are semigroups and f: S→T is an onto homomorphism and I is an ideal of S, then f*(*I*) *is an ideal of T*". She wrote the definition of homomorphism on the left board, wrote "What I know" on the right board, constructed the first-level proof framework, unpacked the conclusion, wrote the second-level proof framework, and decided to do a two-part proof – one part for left ideals and a second part for right ideals. (See Figure 9). With this, Alice seemed to have regained "her footing". At the next meeting, she finished the proof, with some help from us. She continued proving semigroup theorems for the rest of the semester, exhibiting increasing proficiency and self-efficacy.

Discussion

Working with Proof Frameworks

Alice came to us with a reasonable undergraduate mathematics background, some of which she had forgotten. At the first meeting, we explained the use of proof frameworks and our rationale for using them, and she practiced producing several of them. However, at the second meeting she told us that she found this way of working confusing. When she attempted her own alternative method of proving, she got into difficulty, and as a result, was more willing to try using proof frameworks again. Over the course of our subsequent meetings during the first semester, Alice became fluent with writing both first- and second- level proof frameworks and adopted her own methodical way of working. As the first semester went on, she was able to complete proofs with less guidance from us. Indeed, she often mainly required some help with the problem-centered parts of proofs.

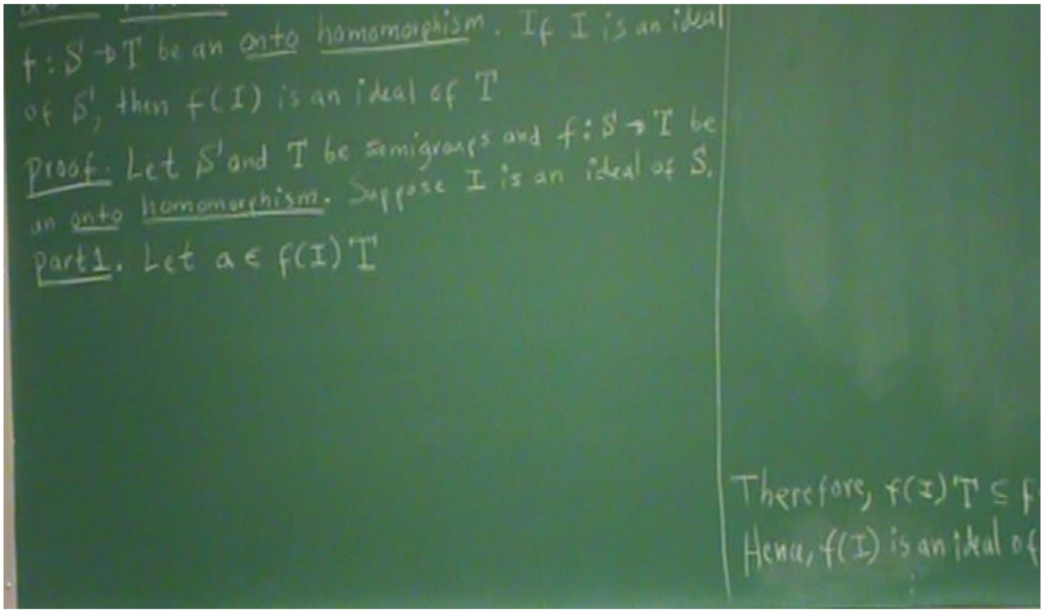

Figure 9. Alice's proof framework for the theorem about homomorphic images of ideals.

During the second semester, Alice continued meeting with us and working on the course notes. She began the second semester with the construction of additional real analysis proofs and seemed to be making very considerable progress, both with writing proof frameworks and with the harder problem-centered parts of proofs. By the end of the real analysis section of the course notes, we felt that she had developed greatly in her proving ability and had developed a sense of self-efficacy (Bandura, 1994, 1995) about proving.

Difficulties Surface When Encountering Unfamiliar Content

However, the subsequent introduction of unfamiliar, abstract content in the form of several definitions and a theorem about semigroups at the 25th meeting seemed to cause her confusion. She constructed only the most obvious examples somewhat hesitantly. Also, when asked to prove the first theorem about semigroups, she did not begin by producing a proof framework, as she had previously consistently done with the real analysis proofs, but rather began writing what she knew or could find in the notes, on the right-hand blackboard. Her proof construction, while not top-down, seemed to consist of first trying to gather as many semigroup ideas as she could, followed by trying to arrange them into a final proof. We feel that concentrating on understanding the unfamiliar abstract content was Alice's initial way of coping, that her working memory may have been overloaded, and that she wasn't able to deal with the additional onus of constructing a proof framework. It was not until the 35th meeting, almost at the end of the second semester, that Alice seemed to have regained her sense of self-efficacy, and she again constructed proofs using the technique of proof frameworks that she had learned and perfected previously with real analysis proofs.

Implications

It seems that coping with newly introduced abstract concepts is not easy, even for someone as experienced as Alice. It also seems that one cannot expect, having learned the skill of constructing proof frameworks in more familiar settings, that this skill will be easily invoked while new abstract content is being learned, perhaps due to working memory overload.

Fragility of Recently Acquired Proving Skills Can Be Overcome

Amongst other things, this case study illustrates the fragility of recently acquired proving skills, in the context of the acquisition of new abstract mathematical concepts. It also suggests that, with persistence, the difficulty due to such fragility can be overcome. Our own experiences as mathematicians suggests that one can (implicitly) learn not to be greatly disturbed by the introduction of several new abstract ideas at once. However, some school curricula avoid certain introductions of concepts, such as the Bourbaki definition of function, because they are considered too abstract (Tabach & Nachlieli, 2015). Further, as Hazzan (1999) found, students sometimes cope by “reducing the level of abstraction.” Yet Alice’s case suggests that, with time, effort and persistence, students can learn to cope with abstraction.

Eventual Successful Use of Proof Frameworks, along with Persistence and Self-efficacy

The initial tendency of many university students to write proofs in a top-down fashion tends to fade after sufficient exposure to writing proof frameworks. One might ask where this tendency comes from. According to Nachlieli and Herbst (2009), it is the norm among U.S. high school geometry teachers to require students, when doing two-column proofs, to follow every statement immediately by a reason. This implies top-down proof construction. However, as noted previously (Selden & Selden, 2013), automating the actions required to write the formal-rhetorical part of a proof (i.e., writing first- and second-level proof frameworks) can allow students to “get started” writing a proof and exposes the “real problem” to be solved in order to complete the proof. For this, persistence and self-efficacy are needed.

References

Baddeley, A. (2000). Short-term and working memory. In E. Tulving & F. I. M. Craik (Eds.), *The Oxford handbook of memory* (pp.77-92). Oxford: Oxford University Press.

Bandura, A. (1994). Self-efficacy. In V. S. Ramachaudran (Ed.), *Encyclopedia of human behavior*, Vol. 4 (pp. 71-81). New York: Academic Press.

Bandura, A. (1995). *Self-efficacy in changing societies*. Cambridge: Cambridge University Press.

Bargh, J. A. (2014). Our unconscious mind. *Scientific American, 310*(1), 30-37.

Benkhalti, A., Selden, A., & Selden, J. (2016) A case study of developing self-efficacy in writing proof frameworks. In T. Fukawa-Connelly, N. Infante, M. Wawro, and S. Brown (Eds.), *Proceedings of the 19th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 542-547). Pittsburgh, Pennsylvania.

Cowan, N. (2001). The magical number 4 in short-term memory: A reconsideration of mental storage capacity. *Behavioral and Brain Sciences, 24,* 87-185.

Ericsson, K. A., & Kintsch, W. (1995). Long-term working memory. *Psychological Review, 102,* 211-245.

Hazzan, O. (1999). Reducing abstraction level when learning abstract algebra concepts. *Educational Studies in Mathematics, 40*(1), 71-90.

Hazzan, O., & Leron, U. (1996). Students' use and misuse of mathematical theorems: The case of Lagrange's Theorem. *For the Learning of Mathematics*, *16*(1), 23-26.

Kalyuga, S. (2014). Managing cognitive load when teaching and learning e-skills. *Proceedings of the e-Skills for Knowledge Production and Innovation Conference* (pp. 155-160). Cape Town, South Africa. Retrieved August 26, 2016 from http://proceedings.e-skillsconference.org/2014/e-skills155-160Kalyuga693.pdf.

Leron, U., & Hazzan, O. (1997). The world according to Johnny: A coping perspective in mathematics education. *Educational Studies in Mathematics*, *32*, 265-292.

Miller, G. A. (1956). The magical number seven, plus or minus two: Some limits on our capacity for processing information. *Psychological Review, 63*, 81-97.

Nachlieli, T., & Herbst, P. (2009). Seeing a colleague encourage a student to make an assumption while proving: What teachers put in play when casting an episode of instruction. *Journal for Research in Mathematics Education, 40,* 427-459.

Pinto, M. M. F., & Tall D. (1999). Student constructions of formal theory: Giving and extracting meaning. In O. Zaslavsky (Ed.), *Proceedings of the 23rd Conference of the International Group for the Psychology of Mathematics Education,* Vol. 3 (pp. 281–288). Haifa, Israel: Israel Institute of Technology.

Savic, M. (2012). *Proof and proving: Logic, impasses, and the relationship to problem solving* (Unpublished doctoral dissertation). New Mexico State University, Las Cruces, New Mexico.

Selden, A., Benkhalti, A., & Selden, J. (2017). The saga of Alice continues: Her progress with proof frameworks evaporates when she encounters unfamiliar concepts, but eventually returns. To appear in *Proceedings of the 20th Annual Conference on Research in Undergraduate Mathematics Education*, which will be available online.

Selden, A., McKee, K., & Selden, J. (2010). Affect, behavioural schemas and the proving process. *International Journal of Mathematical Education in Science and Technology, 41*(2), 199-215.

Selden, A., & Selden, J. (2013). Proof and problem solving at university level. In M. Santos-Trigo & L. Moreno-Armela (Guest. Editor, Special Issue), *The Mathematics Enthusiast, 10*(1&2), 303-334.

Selden, A., & Selden, J. (2008). Overcoming students' difficulties in learning to understand and construct poofs. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics education,* MAA Notes Vol. No. 73 (pp. 95-110). Washington, DC: Mathematical Association of America.

Selden, A., & Selden, J. (1987). Errors and misconceptions in college level theorem proving. In J. D. Novak (Ed.), *Proceedings of the Second International Seminar on Misconceptions and Educational Strategies in Science and Mathematics* (Vol. III, pp. 457-470). Ithaca, NY: Cornell University.

Selden, J., Benkhalti, A., & Selden, A. (2014). An analysis of transition-to-proof course students' proof constructions with a view towards course redesign. In T. Fukawa-Connolly, G. Karakok, K. Keene, & M. Zandieh (Eds.), *Proceedings of the 17th Annual Conference on Research in Undergraduate Mathematics Education* **(**pp. 246-259). Denver, Colorado. Available online at

Selden, J., & Selden, A. (2009). Teaching proving by coordinating aspects of proofs with students' abilities. In D. A. Stylianou, M. L. Blanton, & E. J. Knuth (Eds.), *Teaching and learning proof across grades: A K-16 perspective* (pp. 339-354). New York/Washington, DC: Routledge/National Council of Teachers of Mathematics.

Selden, J., & Selden, A. (1995). Unpacking the logic of mathematical statements. *Educational Studies in Mathematics*, *29*(2), 123-151.

Tabach, M., & Nachlieli, T. (2015). Classroom engagement towards using definitions for developing mathematical objects: The case of function. *Educational Studies in Mathematics*, *90*, 163-187.