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ABSTRACT. Albuquerque and Majid [8] have shown how to view Clifford algebras $Cl_{p,q}$ as twisted group rings whereas Chernov has observed [13] that Clifford algebras can be viewed as images of group algebras of certain 2-groups modulo an ideal generated by a nontrivial central idempotent. Abłamowicz and Fauser [3–5] have introduced a special transposition anti-automorphism of $Cl_{p,q}$, which they called a "transposition", which reduces to reversion in algebras $Cl_{p,0}$ and to conjugation in algebras $Cl_{0,q}$. The purpose of this talk is to bring these concepts together in an attempt to investigate how the algebraic properties of real Clifford algebras, including their periodicity of eight, are a direct consequence of the central product structure of Salingaros vee groups viewed as 2-groups.

Keywords. central product, dihedral group, elementary abelian group, extra-special group, Clifford algebra, Gray code, group algebra, Hopf algebra, quaternionic group, Salingaros vee group, twisted group algebra, Walsh function

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CONTENTS

1. Introduction

The main goal of this survey paper is to show how certain finite groups, in particular, Salingaros vee groups [29–31], and elementary abelian group $(\mathbb{Z}_2)^n = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ (n-times), and their group algebras and twisted groups algebras, arise in the context of Clifford algebras $C\ell_{p,q}$.

Chernov's observation [13] that Clifford algebras $C\ell_{p,q}$ can be viewed as images of (non-twisted) group algebras of suitable 2-groups, conjectured to be Salingaros vee groups [34], allows one to gain a new viewpoint on these algebras and to relate classical group-theoretical results [15,17,24], in particular, on finite 2-groups, to the theory of Clifford algebras. Salingaros classified the groups $G_{p,q}$ – referred to as *Salingaros vee groups* – into five non-isomorphic classes N_{2k-1}, N_{2k}, Ω_{2k-1}, Ω_{2k}, and S_k. These groups, according to the theory of finite 2-groups [15, 24], are central products of extra special groups D_8 – the dihedral group and Q_8 – the quaternionic group, both of order 8, and their centers $\mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, or $\mathbb{Z}_4$. Thus, the properties of these groups and the fact that they fall into the five classes, is reflected by the fact that Clifford algebras $C\ell_{p,q}$ also fall into five isomorphism classes which is well known [14,20,21] and references therein. The structure theorem on these algebras is recalled in Appendix A. Furthermore, the "periodicity of eight" of Clifford algebras viewed as the images of Salingaros vee groups, seems to be related to, if not predicted by, the structure of these groups and their group algebras. Thus, Section 2 is devoted to this approach to Clifford algebras.

Section 3 is devoted to a review of the basic properties of Salingaros vee groups $G_{p,q}$ appearing as finite subgroups of the group of units $C\ell_{p,q}^{\times}$. Furthermore, we will review certain important subgroups of these groups appearing in the context of certain stabilizer groups of primitive idempotents in $C\ell_{p,q}$ [4,5].

Section 4 is devoted to a review of the central product structure of Salingaros vee groups.

In Section 5, we recall how the elementary abelian group $(\mathbb{Z}_2)^n$ appears in the context of defining Clifford product on the set of monomials $\mathbf{e}_{\underline{a}}$ indexed by binary n-tuples $\underline{a}$ from $(\mathbb{Z}_2)^n$. In this first context, Walsh functions – essentially, irreducible characters of $(\mathbb{Z}_2)^n$ – and Gray code – as a certain isomorphism of $(\mathbb{Z}_2)^n$ – are used to define the $C\ell_{p,q}$ algebra product [21, Page 284] and references therein. In particular, a formula given by Lounesto dates back to 1935 and is being attributed to Brauer and Weyl [10]. It will be shown how this formula, applicable only to real Clifford algebras $C\ell_{p,q}$ over quadratic vector spaces (V, Q) with a non-degenerate quadratic form Q of signature (p, q), and for an orthonormal set of basis elements (group generators), can be easily extended to Clifford algebras $C\ell_{p,q,r}$ for degenerate quadratic form Q with $\dim V^{\perp} = r$.

Finally, in Section 6, we briefly recall the group $(\mathbb{Z}_2)^n$ as it appears again in the context of the Clifford algebra $C\ell_{p,q}$ as a twisted group algebra $\mathbb{R}^t[(\mathbb{Z}_2)^n]$ viewed as a Hopf algebra with a certain quasi-triangular structure [8, 16]. This structure is needed to twist the commutative product in the group algebra $\mathbb{R}[(\mathbb{Z}_2)^n]$ in a manner similar to the Brauer and Weyl formula, so that the twisted product is the Clifford product in $C\ell_{p,q}$. It is recalled

that the "transposition" anti-involution of $Cl_{p,q}$ introduced in [3–5] is actually the antipode in the Hopf algebra $\mathbb{R}^t[(\mathbb{Z}_2)^n]$.[1]

Our standard references on the group theory are [15, 17, 27]; in particular, for the theory of p-groups we rely on [24]; for Clifford algebras we use [14, 20, 21] and references therein; on representation theory we refer to [19]; and for the theory of Hopf algebras we refer to [25].

2. Clifford Algebras as Images of Group Algebras

Using Chernov's idea [13], in this section we want to show how Clifford algebras $Cl_{p,q}$ can be viewed as images of group algebras $\mathbb{R}[G]$ of certain 2-groups. It is conjectured [34] that the group G, up to an isomorphism, is the Salingaros vee group $G_{p,q}$ [29–31]. These groups, and their subgroups, have been recently discussed in [4, 5, 11, 22, 23].

Definition 1. Let G be a finite group and let $\mathbb{F}$ be a field[2]. Then the *group algebra* $\mathbb{F}[G]$ is the vector space

$$(1) \qquad \mathbb{F}[G] = \left\{ \sum_{g \in G} \lambda_g g, \ \lambda_g \in \mathbb{F} \right\}$$

with multiplication defined as

$$(2) \qquad \left(\sum_{g \in G} \lambda_g g \right) \left(\sum_{h \in G} \mu_h h \right) = \sum_{g,h \in G} \lambda_g \mu_h (gh) = \sum_{g \in G} \sum_{h \in G} \lambda_h \mu_{h^{-1}g} g$$

where all $\lambda_g, \mu_h \in \mathbb{F}$. [19]

Thus, group algebras are associative unital algebras with the group identity element playing the role of the algebra identity. In the theory of representations of finite groups, all irreducible inequivalent representations are related to a complete decomposition of the group algebra over $\mathbb{C}$ viewed as a *regular* $\mathbb{C}$-module (cf. [19, Maschke Theorem]). The theory is rich on its own. The theory of group characters can then be derived from the representation theory [19], or, as it is often done, from the combinatorial arguments and the theory of characters of the symmetric group [28]. Since in this survey we are only interested in finite groups, we just recall for completeness that every finite group is isomorphic to a subgroup of a symmetric group [27].

We begin by recalling a definition of a p-group.

Definition 2. Let p be a prime. A group G is a *p-group* if every element in G is of order p^k for some $k \geq 1$.

Note that any finite group G of order p^n is a p-group. A classical result states that a center of any p-group is nontrivial, and, by Cauchy's theorem we know that every finite p-group has an element of order p. Thus, in particular, the center of any finite p-group has an element of order p [15, 17, 27]. In the following, we will be working only with finite 2-groups such as, for example, the group $(\mathbb{Z}_2)^n$ and Salingaros vee groups $G_{p,q}$ of order 2^{1+p+q}.

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[1]We remark that twisted group rings can also be described as certain special Ore extensions known as skew polynomial rings [12].

[2]Usually, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ although finite fields are also allowed. In this paper, we will be looking at the real Clifford algebras $Cl_{p,q}$ as images of real group algebras or as real twisted group algebras.

Two important groups in the theory of finite 2-groups and hence in this paper, are the *quaternionic group* Q_8 and the *dihedral group* D_8 (the symmetry group of a square under rotations and reflections), both of order $|Q_8| = |D_8| = 8$. These groups have the following presentations:

Definition 3. The *quaternionic group* Q_8 has the following two presentations:

$$\text{(3a)} \qquad\qquad Q_8 = \langle a, b \mid a^4 = 1, a^2 = b^2, bab^{-1} = a^{-1} \rangle$$

$$\text{(3b)} \qquad\qquad = \langle I, J, \tau \mid \tau^2 = 1, I^2 = J^2 = \tau, IJ = \tau JI \rangle$$

Thus, $Q_8 = \{1, a, a^2, a^3, b, ab, a^2b, a^3b\}$ where the group elements have orders as follows: $|a^2| = 2$, $|a| = |a^3| = |b| = |ab| = |a^2b| = |a^3b| = 4$, so the order structure of Q_8 is $(1, 1, 6)$,[3] and the center $Z(Q_8) = \{1, a^2\} \cong \mathbb{Z}_2$. Here, we can choose $\tau = a^2$. While the presentation (3a) uses only two generators, for convenience and future use, we prefer presentation (3b) which explicitly uses a central element τ of order 2.

Definition 4. The *dihedral group* D_8 (the symmetry group of a square) has the following two presentations:

$$\text{(4a)} \qquad\qquad D_8 = \langle a, b \mid a^4 = b^2 = 1, bab^{-1} = a^{-1} \rangle$$

$$\text{(4b)} \qquad\qquad = \langle \sigma, \tau \mid \sigma^4 = \tau^2 = 1, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$$

Thus, $D_8 = \{1, a, a^2, a^3, b, ab, a^2b, a^3b\}$ where $|a^2| = |b| = |ab| = |a^2b| = |a^3b| = 2$, $|a| = |a^3| = 4$, the order structure of D_8 is $(1, 5, 2)$, and $Z(D_8) = \{1, a^2\} \cong \mathbb{Z}_2$. Here, we can choose $\tau = b$, $\sigma = a$, hence, $\sigma^2 \in Z(D_8)$. That is, σ^2 is our central element of order 2, and our preferred presentation of D_8 is (4b).

In the following two examples, we show how one can construct the Clifford algebra $C\ell_{0,2} \cong \mathbb{H}$ (resp. $C\ell_{1,1}$) as an image of the group algebra of Q_8 (resp. D_8).

Example 1. (Constructing $\mathbb{H} \cong C\ell_{0,2}$ as $\mathbb{R}[Q_8]/\mathcal{J}$)
Define an algebra map ψ from the group algebra $\mathbb{R}[Q_8] \to \mathbb{H} = \operatorname{span}_{\mathbb{R}}\{1, \mathbf{i}, \mathbf{j}, \mathbf{ij}\}$ as follows:

$$\text{(5)} \qquad\qquad 1 \mapsto 1, \quad \tau \mapsto -1, \quad I \mapsto \mathbf{i}, \quad J \mapsto \mathbf{j},$$

Then, $\mathcal{J} = \ker \psi = (1 + \tau)$ for the central element τ of order 2 in Q_8,[4] so $\dim_{\mathbb{R}} \mathcal{J} = 4$ and ψ is surjective. Let $\pi : \mathbb{R}[Q_8] \to \mathbb{R}[Q_8]/\mathcal{J}$ be the natural map $u \mapsto u + \mathcal{J}$. There exists an isomorphism $\varphi : \mathbb{R}[Q_8]/\mathcal{J} \to \mathbb{H}$ such that $\varphi \circ \pi = \psi$ and

$$\pi(I^2) = I^2 + \mathcal{J} = \tau + \mathcal{J} \text{ and } \varphi(\pi(I^2)) = \psi(\tau) = -1 = (\psi(I))^2 = \mathbf{i}^2,$$
$$\pi(J^2) = J^2 + \mathcal{J} = \tau + \mathcal{J} \text{ and } \varphi(\pi(J^2)) = \psi(\tau) = -1 = (\psi(J))^2 = \mathbf{j}^2,$$
$$\pi(IJ + JI) = IJ + JI + \mathcal{J} = (1 + \tau)JI + \mathcal{J} = \mathcal{J} \text{ and}$$
$$\varphi(\pi(IJ + JI)) = \psi(0) = 0 = \psi(I)\psi(J) + \psi(J)\psi(I) = \mathbf{ij} + \mathbf{ji}.$$

Thus, $\mathbb{R}[Q_8]/\mathcal{J} \cong \psi(\mathbb{R}[Q_8]) = \mathbb{H} \cong C\ell_{0,2}$ provided the central element τ is mapped to -1 (see also [13]).

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[3]That is, Q_8 has one element of order 1; one element of order 2; and six elements of order 4.

[4]Here, $(1 + \tau)$ denotes an ideal in $\mathbb{R}[Q_8]$ generated by $1 + \tau$. Note that the two elements $\frac{1}{2}(1 \pm \tau)$ are idempotents which provide an *orthogonal decomposition* of the unity in $\mathbb{R}[Q_8]$.

Example 2. (Constructing $Cl_{1,1}$ as $\mathbb{R}[D_8]/\mathcal{J}$)
Define an algebra map ψ from the group algebra $\mathbb{R}[D_8] \to Cl_{1,1}$ such that:

$$(6) \qquad\qquad 1 \mapsto 1, \quad \tau \mapsto \mathbf{e}_1, \quad \sigma \mapsto \mathbf{e}_2,$$

where $Cl_{1,1} = \mathrm{span}_{\mathbb{R}}\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1\mathbf{e}_2\}$. Then, $\ker\psi = (1 + \sigma^2)$ where σ^2 is a central element of order 2 in D_8. Let $\mathcal{J} = (1 + \sigma^2)$. Thus, $\dim_{\mathbb{R}} \mathcal{J} = 4$ and ψ is surjective. Let $\pi : \mathbb{R}[D_8] \to \mathbb{R}[D_8]/\mathcal{J}$ be the natural map $u \mapsto u + \mathcal{J}$. There exists an isomorphism $\varphi : \mathbb{R}[D_8]/\mathcal{J} \to Cl_{1,1}$ such that $\varphi \circ \pi = \psi$ and

$$\pi(\tau^2) = \tau^2 + \mathcal{J} = 1 + \mathcal{J} \text{ and } \varphi(\pi(\tau^2)) = \psi(1) = 1 = \psi(\tau^2) = (\mathbf{e}_1)^2,$$
$$\pi(\sigma^2) = \sigma^2 + \mathcal{J} \text{ and } \varphi(\pi(\sigma^2)) = \psi(\sigma^2) = -1 = (\mathbf{e}_2)^2,$$
$$\pi(\tau\sigma + \sigma\tau) = \tau\sigma + \sigma\tau + \mathcal{J} = \sigma\tau(1 + \sigma^2) + \mathcal{J} = \mathcal{J} \text{ and}$$
$$\varphi(\pi(\tau\sigma + \sigma\tau) = \psi(0) = 0 = \psi(\tau)\psi(\sigma) + \psi(\sigma)\psi(\tau) = \mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1.$$

Thus, $\mathbb{R}[D_8]/\mathcal{J} \cong Cl_{1,1}$ provided the central element σ^2 is mapped to -1.

It is not difficult to modify Example 2 and construct $Cl_{2,0}$ as the quotient algebra $\mathbb{R}[D_8]/\mathcal{J}$ by changing only the definition of the algebra map ψ given in (6) to

$$(7) \qquad\qquad 1 \mapsto 1, \quad \tau \mapsto \mathbf{e}_1, \quad \sigma \mapsto \mathbf{e}_1\mathbf{e}_2,$$

Then, the rest of Example 2 follows except that of course now $(\mathbf{e}_1)^2 = (\mathbf{e}_2)^2 = 1$. Thus, one can construct $Cl_{2,0}$ as $\mathbb{R}[D_8]/\mathcal{J}$ with again $\mathcal{J} = (1 + \sigma^2)$.

We remark that the fact that can use the group D_8 twice should not come as surprise since $Cl_{1,1} \cong Cl_{2,0}$ (as real Clifford algebras) due to one of the isomorphism theorems stating that $Cl_{p,q} \cong Cl_{q+1,p-1}$, [21, Page 215] (see also [6, 14, 20]) and that we only have, up to an isomorphism, two non-abelian groups of order eight, namely, Q_8 and D_8.

We summarize our two examples as follows. In preparation for Chernov's theorem [13], notice that elements in each group Q_8 and D_8 can be written as follows:

- The quaternionic group Q_8:

$$Q_8 = \{\tau^{\alpha_0} g_1^{\alpha_1} g_2^{\alpha_2} \mid \alpha_k \in \{0, 1\}, \ k = 0, 1, 2\}$$

where $\tau = a^2$ is the central element of order 2 in Q_8, $g_1 = a$, and $g_2 = b$. Thus,

$$(g_1)^2 = a^2 = \tau, \quad (g_2)^2 = b^2 = a^2 = \tau, \quad \tau g_1 g_2 = g_2 g_1.$$

Observe that $|g_1| = |g_2| = 4$ and $\mathbb{R}[Q_8]/\mathcal{J} \cong Cl_{0,2}$ where $\mathcal{J} = (1 + \tau)$.
- The dihedral group D_8:

$$D_8 = \{\tau^{\alpha_0} g_1^{\alpha_1} g_2^{\alpha_2} \mid \alpha_k \in \{0, 1\}, \ k = 0, 1, 2\}$$

where $\tau = a^2$ is the central element of order 2 in D_8, $g_1 = b$, and $g_2 = a$. Thus,

$$(g_1)^2 = b^2 = 1, \quad (g_2)^2 = a^2 = \tau, \quad \tau g_1 g_2 = g_2 g_1.$$

Observe that $|g_1| = 2$, $|g_2| = 4$ and $\mathbb{R}[D_8]/\mathcal{J} \cong Cl_{1,1}$ where $\mathcal{J} = (1 + \tau)$.

Chernov's theorem states the following.

Theorem 1 (Chernov)**.** *Let G be a finite 2-group of order 2^{1+n} generated by a central element τ of order 2 and additional elements $g_1, \ldots, g_n$, which satisfy the following relations:*

(8a) $$\tau^2 = 1, \quad (g_1)^2 = \cdots = (g_p)^2 = 1, \quad (g_{p+1})^2 = \cdots = (g_{p+q})^2 = \tau,$$

(8b) $$\tau g_j = g_j \tau, \quad g_i g_j = \tau g_j g_i, \quad i, j = 1, \ldots, n = p + q,$$

so that $G = \{\tau^{\alpha_0} g_1^{\alpha_1} \cdots g_n^{\alpha_n} \mid \alpha_k \in \{0, 1\}, \ k = 0, 1, \ldots, n\}$. Let $\mathcal{J} = (1 + \tau)$ be an ideal in the group algebra $\mathbb{R}[G]$ and let $C\ell_{p,q}$ be the universal real Clifford algebra generated by $\{\mathbf{e}_k\}, k = 1, \ldots, n = p + q$, where

(9a) $$\mathbf{e}_i^2 = Q(\mathbf{e}_i) \cdot 1 = \varepsilon_i \cdot 1 = \begin{cases} 1 & \text{for } 1 \leq i \leq p; \\ -1 & \text{for } p + 1 \leq i \leq p + q; \end{cases}$$

(9b) $$\mathbf{e}_i \mathbf{e}_j + \mathbf{e}_j \mathbf{e}_i = 0, \quad i \neq j, \quad 1 \leq i, j \leq n.$$

Then, (a) $\dim_{\mathbb{R}} \mathcal{J} = 2^n$; (b) There exists a surjective algebra homomorphism ψ from the group algebra $\mathbb{R}[G]$ to $C\ell_{p,q}$ so that $\ker \psi = \mathcal{J}$ and $\mathbb{R}[G]/\mathcal{J} \cong C\ell_{p,q}$.

Remark 1. Chernov's theorem does not give the existence of the group G. It only states that should such group exist whose generators satisfy relations (8), the result follows. It is not difficult to conjecture that the group G in that theorem is in fact the Salingaros vee group $G_{p,q}$, that is, $\mathbb{R}[G_{p,q}]/\mathcal{J} \cong C\ell_{p,q}$ (see [34]). In fact, we have seen it in Examples 1 and 2 above.

Chernov's theorem. Observe that $G = \{\tau^{\alpha_0} g_1^{\alpha_1} \cdots g_n^{\alpha_n}\} \mid \alpha_k \in \{0, 1\}, \ k = 0, 1, \ldots, n\}$. The existence of a central element τ of order 2 is guaranteed by a well-known fact that the center of any p-group is nontrivial, and by Cauchy Theorem. [27] Define an algebra homomorphism $\psi : \mathbb{R}[G] \to C\ell_{p,q}$ such that

(10) $$1 \mapsto 1, \quad \tau \mapsto -1, \quad g_j \mapsto \mathbf{e}_j, \quad j = 1, \ldots, n.$$

Clearly, $\mathcal{J} \subset \ker \psi$. Let $u \in \mathbb{R}[G]$. Then,

(11) $$u = \sum_{\alpha} \lambda_{\alpha} \tau^{\alpha_0} g_1^{\alpha_1} \cdots g_n^{\alpha_n} = u_1 + \tau u_2$$

where

(12a) $$u_i = \sum_{\widetilde{\alpha}} \lambda_{\widetilde{\alpha}}^{(i)} g_1^{\alpha_1} \cdots g_n^{\alpha_n}, \quad i = 1, 2,$$

(12b) $$\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n) \in \mathbb{R}^{n+1} \quad \text{and} \quad \widetilde{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n.$$

Thus, if $u \in \ker \psi$, then

(13) $$\psi(u) = \sum_{\widetilde{\alpha}} (\lambda_{\widetilde{\alpha}}^{(1)} - \lambda_{\widetilde{\alpha}}^{(2)}) \mathbf{e}_1^{\alpha_1} \cdots \mathbf{e}_n^{\alpha_n} = 0$$

implies $\lambda_{\widetilde{\alpha}}^{(1)} = \lambda_{\widetilde{\alpha}}^{(2)}$ since $\{\mathbf{e}_1^{\alpha_1} \cdots \mathbf{e}_n^{\alpha_n}\}$ is a basis in $C\ell_{p,q}$. Hence,

(14) $$u = (1 + \tau) \sum_{\widetilde{\alpha}} \lambda_{\widetilde{\alpha}}^{(1)} g_1^{\alpha_1} \cdots g_n^{\alpha_n} \in \mathcal{J}.$$

Thus, $\dim_{\mathbb{R}} \ker \psi = 2^n$, $\ker \psi = \mathcal{J}$, $\dim_{\mathbb{R}} \mathbb{R}[G]/\mathcal{J} = 2^{1+n} - 2^n = 2^n$, so ψ is surjective. Let $\varphi : \mathbb{R}[G]/\mathcal{J} \to C\ell_{p,q}$ be such that $\varphi \circ \pi = \psi$ where $\pi : \mathbb{R}[G] \to \mathbb{R}[G]/\mathcal{J}$ is the natural map.

Then, since $\psi(g_j) = \mathbf{e}_j$, $\pi(g_j) = g_j + \mathcal{J}$, we have $\varphi(\pi(g_j)) = \varphi(g_j + \mathcal{J}) = \psi(g_j) = \mathbf{e}_j$ and

$$\pi(g_j)\pi(g_i) + \pi(g_i)\pi(g_j) = (g_j + \mathcal{J})(g_i + \mathcal{J}) + (g_j + \mathcal{J})(g_i + \mathcal{J})$$

(15)
$$= (g_j g_i + g_i g_j) + \mathcal{J} = (1 + \tau)g_j g_i + \mathcal{J} = \mathcal{J}$$

because $g_i g_j = \tau g_j g_i$ in $\mathbb{R}[G]$, τ is central, and $\mathcal{J} = (1+\tau)$. Thus, $g_j + \mathcal{J}, g_i + \mathcal{J}$ anticommute in $\mathbb{R}[G]/\mathcal{J}$ when $i \neq j$. Also,

(16) $\qquad \pi(g_i)\pi(g_i) = (g_i + \mathcal{J})(g_i + \mathcal{J}) = (g_i)^2 + \mathcal{J} = \begin{cases} 1 + \mathcal{J}, & 1 \leq i \leq p; \\ \tau + \mathcal{J}, & p+1 \leq i \leq n; \end{cases}$

due to the relations (8a) on g_i in G. Observe, that

(17) $\qquad \tau + \mathcal{J} = (-1) + (1 + \tau) + \mathcal{J} = (-1) + \mathcal{J}$ in $\mathbb{R}[G]/\mathcal{J}$.

To summarize, the factor algebra $\mathbb{R}[G]/\mathcal{J}$ is generated by the cosets $g_i + \mathcal{J}$ which satisfy these relations:

(18a) $\qquad (g_j + \mathcal{J})(g_i + \mathcal{J}) + (g_j + \mathcal{J})(g_i + \mathcal{J}) = \mathcal{J},$

(18b) $\qquad (g_i)^2 + \mathcal{J} = \begin{cases} 1 + \mathcal{J}, & 1 \leq i \leq p; \\ (-1) + \mathcal{J}, & p+1 \leq i \leq n; \end{cases}$

Thus, the factor algebra $\mathbb{R}[G]/\mathcal{J}$ is a Clifford algebra isomorphic to $C\ell_{p,q}$ provided $\mathcal{J} = (1+\tau)$ for the central element τ of order 2 in G. $\qquad \square$

3. Salingaros Vee Groups $G_{p,q} \subset C\ell_{p,q}^\times$

Let $\dim_\mathbb{R} V = n$ and Q be a non-degenerate quadratic form on V:

(19) $\qquad Q(\mathbf{x}) = \varepsilon_1 x_1^2 + \varepsilon_2 x_2^2 + \cdots + \varepsilon_n x_n^2,$

$\varepsilon_i = \pm 1$ and $\mathbf{x} = x_1 \mathbf{e}_1 + \cdots + x_n \mathbf{e}_n \in V$ for an orthonormal basis $\mathcal{B}_1 = \{\mathbf{e}_i, 1 \leq i \leq n\}$. Q has an arbitrary signature $-n \leq p - q \leq n$ where p (resp. q) denotes the number of $+1$'s (resp. -1's) in (19), and $p + q = n$. Let $C\ell_{p,q}$ be the universal Clifford algebra of (V, Q) obtained, for example, via Chevalley's construction [21, Chapter 22].

Then, let $\mathcal{B}$ be the canonical basis of $\bigwedge V$ generated by $\mathcal{B}_1$, $[n] = \{1, 2, \ldots, n\}$ and denote arbitrary, canonically ordered subsets of $[n]$, by underlined Roman characters. The basis elements of $\bigwedge V$, or, of $C\ell_{p,q}$ due to the linear space isomorphism $\bigwedge V \to C\ell_{p,q}$, can be indexed by these finite ordered subsets as $\mathbf{e}_{\underline{i}} = \wedge_{i \in \underline{i}} \mathbf{e}_i$.

Now, let $G_{p,q}$ be a finite group in any real Clifford algebra $C\ell_{p,q}$ (simple or semisimple) with a with a binary operation being just the Clifford product, namely:

(20) $\qquad G_{p,q} = \{\pm \mathbf{e}_{\underline{i}} \mid \mathbf{e}_{\underline{i}} \in \mathcal{B} \text{ with } \mathbf{e}_{\underline{i}}\mathbf{e}_{\underline{j}} \text{ denoting the Clifford product}\}.$

Thus, $G_{p,q}$ may be presented as follows:

(21) $\qquad G_{p,q} = \langle -1, \mathbf{e}_1, \ldots, \mathbf{e}_n \mid \mathbf{e}_i \mathbf{e}_j = -\mathbf{e}_j \mathbf{e}_i \text{ for } i \neq j \text{ and } \mathbf{e}_i^2 = \pm 1 \rangle,$

where $\mathbf{e}_i^2 = 1$ for $1 \leq i \leq p$ and $\mathbf{e}_i^2 = -1$ for $p + 1 \leq i \leq n = p + q$. In the following, the elements $\mathbf{e}_{\underline{i}} = \mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_k}$ will be denoted for short as $\mathbf{e}_{i_1 i_2 \cdots i_k}$ for $k \geq 1$ while $\mathbf{e}_\emptyset$ will be denoted as 1, the identity element of $G_{p,q}$ (and $C\ell_{p,q}$).

This 2-group of order $2 \cdot 2^{p+q} = 2^{n+1}$ is known as *Salingaros vee group* and it has been discussed, for example, by Salingaros [29–31], Varlamov [32, 33], Helmstetter [18],

Abłamowicz and Fauser [4, 5], Maduranga and Abłamowicz [23], and most recently by Brown [11]. We should recall here that $G_{p,q}$ is a discrete subgroup of $\mathbf{Pin}(p,q) \subset \mathbf{\Gamma}_{p,q}$ (Lipschitz group) (Lounesto [21]).

In preparation for discussing properties of the groups $G_{p,q}$ and related to them subgroups, we recall a definition of the derived subgroup $G' \subset G$ and a proposition that gives some of its properties [27].

Definition 5. If G is a group and $x, y \in G$, then their *commutator* $[x, y]$ is the element $xyx^{-1}y^{-1}$. If X and Y are subgroups of G, then the *commutator subgroup* $[X, Y]$ of G is defined by

$$(22) \qquad [X, Y] = \langle [x, y] \mid x \in X, y \in Y \rangle,$$

that is, the group $[X, Y]$ is generated by all the commutators $[x, y]$. In particular, the *derived subgroup* G' of G is defined as $G' = [G, G]$.

Proposition 1. *Let G be a group.*

    (i) *G' is a normal subgroup of G, and G/G' is abelian.*
    (ii) *If H is a normal subgroup of G and G/H is abelian, then $G' \subseteq H$.*

3.1. **Transposition Anti-Involution in $C\ell_{p,q}$.** Let us now recall a definition and some of its basic properties of a special anti-involution $T_{\tilde{\varepsilon}}$ in a Clifford algebra $C\ell_{p,q}$ referred to as "transposition". This anti-involution was introduced in [3–5] where its properties were investigated at length. In particular, it allowed for an introduction of a reciprocal basis in a Clifford algebra $C\ell_{p,q}$ and, subsequently, a new spinor product on spinor spaces, and a classification of its (infinite) groups of invariance. In the following, we limit ourselves only to reviewing certain finite groups appearing in this context.

Definition 6. The *transposition* $T_{\tilde{\varepsilon}}$ of $C\ell_{p,q}$ is defined as:

$$(23) \qquad T_{\tilde{\varepsilon}} : C\ell_{p,q} \to C\ell_{p,q}, \qquad \sum_{\underline{i} \in 2^{[n]}} u_{\underline{i}} \mathbf{e}_{\underline{i}} \mapsto \sum_{\underline{i} \in 2^{[n]}} u_{\underline{i}} (\mathbf{e}_{\underline{i}})^{-1}$$

It is the *antipode* map S known from the theory of group algebras $\mathbb{F}[G]$

$$(24) \qquad \mathbb{F}[G] \to \mathbb{F}[G], \qquad \sum_{g \in G} \lambda_g g \mapsto \sum_{g \in G} \lambda_g g^{-1}$$

viewed as Hopf algebras [25]. Here are a few of its properties and a few finite related groups. For more details, see [3–5].

- $T_{\tilde{\varepsilon}}$ is an anti-involution of $C\ell_{p,q}$ which reduces to reversion in $C\ell_{p,0}$ and to conjugation in $C\ell_{0,q}$.
- Depending on the value of $(p - q) \bmod 8$, where (p, q) is the signature of Q, $T_{\tilde{\varepsilon}}$ gives rise to transposition, Hermitian complex, and Hermitian quaternionic conjugation of spinor representation matrices.
- $T_{\tilde{\varepsilon}}(\mathbf{e}_{\underline{i}}) = \mathbf{e}_{\underline{i}}^{-1}$ hence $T_{\tilde{\varepsilon}}(\mathbf{e}_{\underline{i}}) = \mathbf{e}_{\underline{i}}$ (resp. $T_{\tilde{\varepsilon}}(\mathbf{e}_{\underline{i}}) = -\mathbf{e}_{\underline{i}}$) when $(\mathbf{e}_{\underline{i}})^2 = 1$ (resp. $(\mathbf{e}_{\underline{i}})^2 = -1$) (elements of order 2 and 4, respectively, in $G_{p,q}$).
- $T_{\tilde{\varepsilon}}(f) = f$ for any primitive idempotent f.
- Let $S = C\ell_{p,q}f$ be a spinor (minimal left) ideal in a simple algebra $C\ell_{p,q}$ generated by a primitive idempotent f. Then, $T_{\tilde{\varepsilon}}$ defines a dual spinor space $S^* = T_{\tilde{\varepsilon}}(S)$

and a $\mathbb{K}$-valued, where $\mathbb{K} = fC\ell_{p,q}f$, spinor norm $(\psi, \phi) = T_{\tilde{\varepsilon}}(\psi)\phi$ on S invariant under (infinite) group $G^{\varepsilon}_{p,q}$ (with $G_{p,q} < G^{\varepsilon}_{p,q}$) different, in general, from spinor norms related to reversion and conjugation in $C\ell_{p,q}$.

- $G_{p,q}$ act transitively on a complete set $\mathcal{F}$, $|\mathcal{F}| = 2^{q-r_{q-p}}$, of mutually annihilating primitive idempotents where r_i is the Radon-Hurwitz number. See a footnote in Appendix A for a definition of r_i.
- The normal stabilizer subgroup $G_{p,q}(f) \triangleleft G_{p,q}$ of f is of order $2^{1+p+r_{q-p}}$ and monomials m_i in its (non-canonical) left transversal together with f determine a spinor basis in S.
- The stabilizer groups $G_{p,q}(f)$ and the invariance groups $G^{\varepsilon}_{p,q}$ of the spinor norm have been classified according to the signature (p, q) for $(p + q) \leq 9$ in simple and semisimple algebras $C\ell_{p,q}$.
- $G_{p,q}$ permutes the spinor basis elements modulo the commutator subgroup $G'_{p,q}$ by left multiplication.
- The ring $\mathbb{K} = fC\ell_{p,q}f$ is $G_{p,q}$-invariant.

3.2. Important Finite Subgroups of $C\ell^{\times}_{p,q}$.

In this section, we summarize properties and definitions of some finite subgroups of the group of invertible elements $C\ell^{\times}_{p,q}$ in the Clifford algebra $C\ell_{p,q}$. These groups were defined in [3–5].

- $G_{p,q}$ – Salingaros vee group of order $|G_{p,q}| = 2^{1+p+q}$,
- $G'_{p,q} = \{1, -1\}$ – the commutator subgroup of $G_{p,q}$,
- Let $\mathcal{O}(f)$ be the orbit of f under the conjugate action of $G_{p,q}$, and let $G_{p,q}(f)$ be the stabilizer of f. Let

(25) $$N = |\mathcal{F}| = [G_{p,q} : G_{p,q}(f)] = |\mathcal{O}(f)| = |G_{p,q}|/|G_{p,q}(f)| = 2 \cdot 2^{p+q}/|G_{p,q}(f)|$$

then $N = 2^k$ (resp. $N = 2^{k-1}$) for simple (resp. semisimple) $C\ell_{p,q}$ where $k = q - r_{q-p}$ and $[G_{p,q} : G_{p,q}(f)]$ is the index of $G_{p,q}(f)$ in $G_{p,q}$.
- $G_{p,q}(f) \triangleleft G_{p,q}$ and $|G_{p,q}(f)| = 2^{1+p+r_{q-p}}$ (resp. $|G_{p,q}(f)| = 2^{2+p+r_{q-p}}$) for simple (resp. semisimple) $C\ell_{p,q}$.
- The set of commuting monomials $\mathcal{T} = \{\mathbf{e}_{\underline{i}_1}, \ldots, \mathbf{e}_{\underline{i}_k}\}$ (squaring to 1) in the primitive idempotent $f = \frac{1}{2}(1 \pm \mathbf{e}_{\underline{i}_1}) \cdots \frac{1}{2}(1 \pm \mathbf{e}_{\underline{i}_k})$ is point-wise stabilized by $G_{p,q}(f)$.
- $T_{p,q}(f) := \langle \pm 1, \mathcal{T} \rangle \cong G'_{p,q} \times \langle \mathbf{e}_{\underline{i}_1}, \ldots, \mathbf{e}_{\underline{i}_k} \rangle \cong G'_{p,q} \times (\mathbb{Z}_2)^k$, the *idempotent group* of f with $|T_{p,q}(f)| = 2^{1+k}$,
- $K_{p,q}(f) = \langle \pm 1, m \mid m \in \mathcal{K} \rangle < G_{p,q}(f)$ – the *field group* of where f is a primitive idempotent in $C\ell_{p,q}$, $\mathbb{K} = fC\ell_{p,q}f$, and $\mathcal{K}$ is a set of monomials (a transversal) in $\mathcal{B}$ which span $\mathbb{K}$ as a real algebra. Thus,

(26) $$|K_{p,q}(f)| = \begin{cases} 2, & p - q = 0, 1, 2 \bmod 8; \\ 4, & p - q = 3, 7 \bmod 8; \\ 8, & p - q = 4, 5, 6 \bmod 8. \end{cases}$$

- $G^{\varepsilon}_{p,q} = \{g \in C\ell_{p,q} \mid T_{\tilde{\varepsilon}}(g)g = 1\}$ (infinite group)

Before we state the main theorem from [5] that relates the above finite groups to the Salingaros vee groups, we recall the definition of a *transversal*.

Definition 7. Let K be a subgroup of a group G. A *transversal* ℓ of K in G is a subset of G consisting of exactly one element $\ell(bK)$ from every (left) coset bK, and with $\ell(K) = 1$.

Theorem 2 (Main Theorem). *Let f be a primitive idempotent in $C\ell_{p,q}$ and let $G_{p,q}$, $G_{p,q}(f)$, $T_{p,q}(f)$, $K_{p,q}(f)$, and $G'_{p,q}$ be the groups defined above. Let $S = C\ell_{p,q}f$ and $\mathbb{K} = fC\ell_{p,q}f$.*

(i) *Elements of $T_{p,q}(f)$ and $K_{p,q}(f)$ commute.*

(ii) *$T_{p,q}(f) \cap K_{p,q}(f) = G'_{p,q} = \{\pm 1\}$.*

(iii) *$G_{p,q}(f) = T_{p,q}(f)K_{p,q}(f) = K_{p,q}(f)T_{p,q}(f)$.*

(iv) *$|G_{p,q}(f)| = |T_{p,q}(f)K_{p,q}(f)| = \frac{1}{2}|T_{p,q}(f)||K_{p,q}(f)|$.*

(v) *$G_{p,q}(f) \lhd G_{p,q}$, $T_{p,q}(f) \lhd G_{p,q}$, and $K_{p,q}(f) \lhd G_{p,q}$. In particular, $T_{p,q}(f)$ and $K_{p,q}(f)$ are normal subgroups of $G_{p,q}(f)$.*

(vi) *We have:*

$$G_{p,q}(f)/K_{p,q}(f) \cong T_{p,q}(f)/G'_{p,q}, \tag{27}$$

$$G_{p,q}(f)/T_{p,q}(f) \cong K_{p,q}(f)/G'_{p,q}. \tag{28}$$

(vii) *We have:*

$$(G_{p,q}(f)/G'_{p,q})/(T_{p,q}(f)/G'_{p,q}) \cong G_{p,q}(f)/T_{p,q}(f) \cong K_{p,q}(f)/\{\pm 1\} \tag{29}$$

and the transversal of $T_{p,q}(f)$ in $G_{p,q}(f)$ spans $\mathbb{K}$ over $\mathbb{R}$ modulo f.

(viii) *The transversal of $G_{p,q}(f)$ in $G_{p,q}$ spans S over $\mathbb{K}$ modulo f.*

(ix) *We have $(G_{p,q}(f)/T_{p,q}(f)) \lhd (G_{p,q}/T_{p,q}(f))$ and*

$$(G_{p,q}/T_{p,q}(f))/(G_{p,q}(f)/T_{p,q}(f)) \cong G_{p,q}/G_{p,q}(f) \tag{30}$$

and the transversal of $T_{p,q}(f)$ in $G_{p,q}$ spans S over $\mathbb{R}$ modulo f.

(x) *The stabilizer $G_{p,q}(f)$ can be viewed as*

$$G_{p,q}(f) = \bigcap_{x \in T_{p,q}(f)} C_{G_{p,q}}(x) = C_{G_{p,q}}(T_{p,q}(f)) \tag{31}$$

where $C_{G_{p,q}}(x)$ is the centralizer of x in $G_{p,q}$ and $C_{G_{p,q}}(T_{p,q}(f))$ is the centralizer of $T_{p,q}(f)$ in $G_{p,q}$.

3.3. **Summary of Some Basic Properties of Salingaros Vee Groups $G_{p,q}$.** In the following, we summarize some basic properties of Salingaros vee groups $G_{p,q}$.

- $|G_{p,q}| = 2^{1+p+q}$, $|G'_{p,q}| = 2$ because $G'_{p,q} = \{\pm 1\}$,
- When $p + q \geq 1$, $G_{p,q}$ is not simple as it has a nontrivial normal subgroup of order 2^m for every $m < 1 + p + q$ (because every p-group of order p^n has a normal subgroup of order p^m for every $m \neq n$).
- When $p + q \geq 1$, the center of any group $G_{p,q}$ is non-trivial since $2 \mid |Z(G_{p,q})|$ and so every group $G_{p,q}$ has a central element τ of order 2. It is well-known that for any prime p and a finite p-group $G \neq \{1\}$, the center of G is non-trivial (Rotman [27]).
- Every element of $G_{p,q}$ is of order 1, 2, or 4.
- Since $[G_{p,q} : G'_{p,q}] = |G_{p,q}|/|G'_{p,q}| = 2^{p+q}$, each $G_{p,q}$ has 2^{p+q} linear characters (James and Liebeck [19]).
- The number N of conjugacy classes in $G_{p,q}$, hence, the number of irreducible inequivalent representations of $G_{p,q}$, is $1 + 2^{p+q}$ (resp. $2 + 2^{p+q}$) when $p + q$ is even (resp. odd) (Maduranga [22]).
- We have the following result (see also Varlamov [33]):

Theorem 3. *Let $G_{p,q} \subset C\ell_{p,q}$. Then,*

(32)
$$Z(G_{p,q}) = \begin{cases} \{\pm 1\} \cong \mathbb{Z}_2 & \text{if } p - q \equiv 0, 2, 4, 6 \pmod 8; \\ \{\pm 1, \pm \beta\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 & \text{if } p - q \equiv 1, 5 \pmod 8; \\ \{\pm 1, \pm \beta\} \cong \mathbb{Z}_4 & \text{if } p - q \equiv 3, 7 \pmod 8. \end{cases}$$

as a consequence of $Z(C\ell_{p,q}) = \{1\}$ (resp. $\{1, \beta\}$) when $p + q$ is even resp. odd) where $\beta = \mathbf{e}_1 \mathbf{e}_2 \cdots \mathbf{e}_n$, $n = p + q$, is the unit pseudoscalar in $C\ell_{p,q}$.

- In Salingaros' notation, the five isomorphism classes denoted as $N_{2k-1}, N_{2k}, \Omega_{2k-1}, \Omega_{2k}, S_k$ correspond to our notation $G_{p,q}$ as follows:

TABLE 1. Vee groups $G_{p,q}$ in Clifford algebras $C\ell_{p,q}$

Group	Center	Group order	$\dim_{\mathbb{R}} C\ell_{p,q}$
N_{2k-1}	$\mathbb{Z}_2$	2^{2k+1}	2^{2k}
N_{2k}	$\mathbb{Z}_2$	2^{2k+1}	2^{2k}
Ω_{2k-1}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	2^{2k+2}	2^{2k+1}
Ω_{2k}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	2^{2k+2}	2^{2k+1}
S_k	$\mathbb{Z}_4$	2^{2k+2}	2^{2k+1}

$$N_{2k-1} \leftrightarrow G_{p,q} \subset C\ell_{p,q}, \ p - q \equiv 0, 2 \pmod 8, \ \mathbb{K} \cong \mathbb{R};$$
$$N_{2k} \leftrightarrow G_{p,q} \subset C\ell_{p,q}, \ p - q \equiv 4, 6 \pmod 8, \ \mathbb{K} \cong \mathbb{H};$$
$$\Omega_{2k-1} \leftrightarrow G_{p,q} \subset C\ell_{p,q}, \ p - q \equiv 1 \pmod 8, \ \mathbb{K} \cong \mathbb{R} \oplus \mathbb{R};$$
$$\Omega_{2k} \leftrightarrow G_{p,q} \subset C\ell_{p,q}, \ p - q \equiv 5 \pmod 8, \ \mathbb{K} \cong \mathbb{H} \oplus \mathbb{H};$$
$$S_k \leftrightarrow G_{p,q} \subset C\ell_{p,q}, \ p - q \equiv 3, 7 \pmod 8, \ \mathbb{K} \cong \mathbb{C}.$$

(Salingaros [29–31], Brown [11], Varlamov [33])

The first few vee groups $G_{p,q}$ of low orders $4, 8, 16$ corresponding to Clifford algebras $C\ell_{p,q}$ in dimensions $p + q = 1, 2, 3$, are:

Groups of order 4: $\quad G_{1,0} = D_4, \quad G_{0,1} = \mathbb{Z}_4,$

Groups of order 8: $\quad G_{2,0} = D_8 = N_1, \quad G_{1,1} = D_8 = N_1, \quad G_{0,2} = Q_8 = N_2,$

Groups of order 16: $\quad G_{3,0} = S_1, \quad G_{2,1} = \Omega_1, \quad G_{1,2} = S_1, \quad G_{0,3} = \Omega_2.$

where D_8 is the dihedral group of a square, Q_8 is the quaternionic group, and $D_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. For a construction of inequivalent irreducible representations and characters of these groups see Maduranga and Abłamowicz [23], and Maduranga [22].

4. CENTRAL PRODUCT STRUCTURE OF $G_{p,q}$

We recall first a few definitions and results pertaining to finite p-groups that will be needed in the sequel.

Definition 8 (Gorenstein [17]). A finite abelian p-group is *elementary abelian* if every non-trivial element has order p.

Example 3. $(D_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$ is elementary abelian)
$(\mathbb{Z}_p)^k = \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$ (k-times), in particular, $\mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, etc, are elementary abelian.

Definition 9 (Dornhoff [15]). A finite p-group P is *extra-special* if (i) $P' = Z(P)$, (ii) $|P'| = p$, and (iii) P/P' is elementary abelian.

Example 4. (D_8 is extra-special)
$D_8 = \langle a, b \mid a^4 = b^2 = 1, bab^{-1} = a^{-1} \rangle$ is extra-special because:

- $Z(D_8) = D_8' = [D_8, D_8] = \langle a^2 \rangle$, $|Z(D_8)| = 2$,
- $D_8/D_8' = D_8/Z(D_8) = \langle \langle a^2 \rangle, a\langle a^2 \rangle, b\langle a^2 \rangle, ab\langle a^2 \rangle \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Example 5. (Q_8 is extra-special)
$Q_8 = \langle a, b \mid a^4 = 1, a^2 = b^2, bab^{-1} = a^{-1} \rangle$ is extra-special because:

- $Z(Q_8) = Q_8' = [Q_8, Q_8] = \langle a^2 \rangle$, $|Z(Q_8)| = 2$,
- $Q_8/Q_8' = Q_8/Z(Q_8) = \langle \langle a^2 \rangle, a\langle a^2 \rangle, b\langle a^2 \rangle, ab\langle a^2 \rangle \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Let us recall now definitions of internal and external central products of groups.

Definition 10 (Gorenstein [17]).

(1) A group G is an *internal central product* of two subgroups H and K if:
   (a) $[H, K] = \langle 1 \rangle$;
   (b) $G = HK$;
(2) A group G is an *external central product* $H \circ K$ of two groups H and K with $H_1 \leq Z(H)$ and $K_1 \leq Z(K)$ if there exists an isomorphism $\theta : H_1 \to K_1$ such that G is $(H \times K)/N$ where
$$N = \{(h, \theta(h^{-1})) \mid h \in H_1\}.$$
   Clearly: $N \lhd (H \times K)$ and $|H \circ K| = |H||K|/|N| \leq |H \times K| = |H||K|$.

Here we recall an important result on extra-special p-groups as central products.

Lemma 1 (Leedham-Green and McKay [24]). *An extra-special p-group has order p^{2n+1} for some positive integer n, and is the iterated central product of non-abelian groups of order p^3.*

As a consequence, we have the following lemma and a corollary. For their proofs, see [11].

Lemma 2. $Q_8 \circ Q_8 \cong D_8 \circ D_8 \not\cong D_8 \circ Q_8$, *where D_8 is the dihedral group of order 8 and Q_8 is the quaternion group.*

Corollary 1.

- $G_{3,1} \cong D_8 \circ D_8 \cong Q_8 \circ Q_8$,
- $G_{4,0} \cong D_8 \circ Q_8 \cong Q_8 \circ D_8$.

The following theorem is of critical importance for understanding the central product structure of Salingaros vee groups.

Theorem 4 (Leedham-Green and McKay [24]). *There are exactly two isomorphism classes of extra-special groups of order 2^{2n+1} for positive integer n. One isomorphism type arises as the iterated central product of n copies of D_8; the other as the iterated central product of n groups isomorphic to D_8 and Q_8, including at least one copy of Q_8. That is,*

1: $D_8 \circ D_8 \circ \cdots \circ D_8 \circ D_8$, *or,*

2: $D_8 \circ D_8 \circ \cdots \circ D_8 \circ Q_8$.

where it is understood that these are iterated central products; that is, $D_8 \circ D_8 \circ D_8$ is really $(D_8 \circ D_8) \circ D_8$ and so on.

Thus, the above theorem now explains the following theorem due to Salingaros regarding the iterative central product structure of the finite 2-groups named after him.

Theorem 5 (Salingaros Theorem [31])**.** *Let $N_1 = D_8$, $N_2 = Q_8$, and $(G)^{\circ k}$ be the iterative central product $G \circ G \circ \cdots \circ G$ (k times) of G. Then, for $k \geq 1$:*

(1) $N_{2k-1} \cong (N_1)^{\circ k} = (D_8)^{\circ k}$,
(2) $N_{2k} \cong (N_1)^{\circ k} \circ N_2 = (D_8)^{\circ(k-1)} \circ Q_8$,
(3) $\Omega_{2k-1} \cong N_{2k-1} \circ (\mathbb{Z}_2 \times \mathbb{Z}_2) = (D_8)^{\circ k} \circ (\mathbb{Z}_2 \times \mathbb{Z}_2)$,
(4) $\Omega_{2k} \cong N_{2k} \circ (\mathbb{Z}_2 \times \mathbb{Z}_2) = (D_8)^{\circ(k-1)} \circ Q_8 \circ (\mathbb{Z}_2 \times \mathbb{Z}_2)$,
(5) $S_k \cong N_{2k-1} \circ \mathbb{Z}_4 \cong N_{2k} \circ \mathbb{Z}_4 = (D_8)^{\circ k} \circ \mathbb{Z}_4 \cong (D_8)^{\circ(k-1)} \circ Q_8 \circ \mathbb{Z}_4$.

In the above theorem:

- $\mathbb{Z}_2$, $\mathbb{Z}_4$ are cyclic groups of order 2 and 4, respectively;
- D_8 and Q_8 are the dihedral group of a square and the quaternionic group;
- $\mathbb{Z}_2 \times \mathbb{Z}_2$ is elementary abelian of order 4;
- N_{2k-1} and N_{2k} are extra-special of order 2^{2k+1}; e.g., $N_1 = D_8$ and $N_2 = Q_8$;
- $\Omega_{2k-1}, \Omega_{2k}, S_k$ are of order 2^{2k+2}.
- $\circ$ denotes the iterative central product of groups with, e.g., $(D_8)^{\circ k}$ denotes the iterative central product of k-copies of D_8, etc.,

We can tabulate the above results for Salingaros vee groups $G_{p,q}$ of orders ≤ 256, $(p+q \leq 7)$ (Brown [11]) in the following table:

TABLE 2. Salingaros Vee Groups $|G_{p,q}| \leq 256$

Isomorphism Class	Salingaros Vee Groups
N_{2k}	$N_0 \cong G_{0,0},\ N_2 \cong Q_8 \cong G_{0,2},\ N_4 \cong G_{4,0},\ N_6 \cong G_{6,0}$
N_{2k-1}	$N_1 \cong D_8 \cong G_{2,0},\ N_3 \cong G_{3,1},\ N_5 \cong G_{0,6}$
Ω_{2k}	$\Omega_0 \cong G_{1,0},\ \Omega_2 \cong G_{0,3},\ \Omega_4 \cong G_{5,0},\ \Omega_6 \cong G_{6,1}$
Ω_{2k-1}	$\Omega_1 \cong G_{2,1},\ \Omega_3 \cong G_{3,2},\ \Omega_5 \cong G_{0,7}$
S_k	$S_0 \cong G_{0,1},\ S_1 \cong G_{3,0},\ S_2 \cong G_{4,1},\ S_3 \cong G_{7,0}$

5. Clifford Algebras Modeled with Walsh Functions

Until now, the finite 2-groups such as the Salingaros vee groups $G_{p,q}$ have appeared either as finite subgroups of the group of units $C\ell_{p,q}^{\times}$ in the Clifford algebra, or, as groups whose group algebra modulo a certain ideal generated by $1 + \tau$ for some central element τ of order 2 was isomorphic to the given Clifford algebra $C\ell_{p,q}$. In these last two sections, we

recall how the (elementary abelian) group $(\mathbb{Z}_2)^n$ can be used to define a Clifford product on a suitable vector space.

In this section, we recall the well-known construction of the Clifford product on the set of monomial terms $\mathbf{e}_{\underline{a}}$ indexed by binary n-tuples $\underline{a} \in (\mathbb{Z}_2)^n$, which, when extended by linearity, endows the set with the structure of the Clifford algebra $C\ell_{p,q}$. This approach is described in Lounesto [21, Chapter 21]. We will show how it can be extended to Clifford algebras $C\ell_{p,q,r}$ over (real) quadratic vector spaces with degenerate quadratic forms.

In the last section we will briefly mention the approach of Albuquerque and Majid [25] in which the Clifford algebra structure is introduced in a suitably twisted group algebra $\mathbb{R}^t[(\mathbb{Z}_2)^n]$ using Hopf algebraic methods.

Let $\mathcal{B}^n = \{\underline{a} = a_1 a_2 \ldots a_n \mid a_i = 0, 1, \ \underline{a} \oplus \underline{b} = \underline{c} \text{ as } c_i = a_i + b_i \bmod 2\}$ be a group of binary n-tuples with addition $\oplus$, that is, $\mathcal{B}^n \cong (\mathbb{Z}_2)^n$.

Definition 11 (Walsh function). A *Walsh function* $w_{\underline{a}}$ indexed by $\underline{a} \in \mathcal{B}^n$ is a function from $\mathcal{B}^n$ to the multiplicative group $\{\pm 1\}$ defined as

(33) $$w_{\underline{a}}(\underline{b}) = (-1)^{\sum_{i=1}^n a_i b_i} = \pm 1, \quad \underline{a}, \underline{b} \in \mathcal{B}^n,$$

which satisfies $w_{\underline{k}}(\underline{a} \oplus \underline{b}) = w_{\underline{k}}(\underline{a}) w_{\underline{k}}(\underline{b})$ and $w_{\underline{a}}(\underline{b}) = w_{\underline{b}}(\underline{a})$.

Observe that the first condition on $w_{\underline{a}}$ simply states that each $w_{\underline{a}}$ is a group homomorphism from $\mathcal{B}^n$ to the group $\{\pm 1\}$.

Definition 12 (Gray code). A *Gray code* $g : \mathcal{B}^n \to \mathcal{B}^n$ with the property $g(\underline{a} \oplus \underline{b}) = g(\underline{a}) \oplus g(\underline{b})$ is defined as

(34) $$g(\underline{k})_1 = k_1, \quad g(\underline{k})_i = k_{i-1} + k_i \bmod 2, \quad i = 2, \ldots, n.$$

Thus, g is a group isomorphism which reorders Walsh functions into a *sequency order* with a *single digit change code* [21, Section 21.2, page 281].

Given that the Gray code g is an isomorphism, Lounesto defines its inverse $h : \mathcal{B}^n \to \mathcal{B}^n$ as

(35) $$h(\underline{a})_i = \sum_{j=1}^i a_j \bmod 2.$$

Now, take an $\mathbb{R}$-vector space $\mathcal{A}$ with a basis consisting of 2^n elements $\mathbf{e}_{\underline{a}}$ labeled by the binary n-tuples $\underline{a} = a_1 a_2 \ldots a_n$ as

(36) $$\mathbf{e}_{\underline{a}} = \mathbf{e}_1^{a_1} \mathbf{e}_2^{a_2} \cdots \mathbf{e}_n^{a_n}, \quad a_i = 0, 1;$$

for some n symbols $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$, and define an algebra product on $\mathcal{A}$ which on the basis elements $\mathbf{e}_{\underline{a}}$ is computed as follows:

(37) $$\mathbf{e}_{\underline{a}} \mathbf{e}_{\underline{b}} = (-1)^{\sum_{i=1}^p a_i b_i} w_{\underline{a}}(h(\underline{b})) \mathbf{e}_{\underline{a} \oplus \underline{b}},$$

for some $1 \le p \le n$. Then, together with this product, $\mathcal{A}$ becomes the Clifford algebra $C\ell_{p,q}$, where $q = n - p$, of a non-degenerate quadratic form Q of signature (p, q). See Lounesto [21, Page 284] and his reference to (37) as the formula of Brauer and Weyl from 1935 [10].

Remark 2. Observe that if the scalar factor in front of $\mathbf{e}_{\underline{a}\oplus\underline{b}}$ in (37) were set to be identically equal to 1, then we would have $\mathbf{e}_{\underline{a}}\mathbf{e}_{\underline{b}} = \mathbf{e}_{\underline{b}}\mathbf{e}_{\underline{a}}$ for any $\mathbf{e}_{\underline{a}}, \mathbf{e}_{\underline{b}} \in \mathcal{A}$. Thus, the algebra $\mathcal{A}$ would be isomorphic to the (abelian) group algebra $\mathbb{R}[G]$ where $G \cong (\mathbb{Z}_2)^n$. That is, the scalar factor introduces a twist in the algebra product in $\mathcal{A}$ and so it makes $\mathcal{A}$, hence the Clifford algebra $C\ell_{p,q}$, isomorphic to the twisted group algebra $\mathbb{R}^t[(\mathbb{Z}_2)^n]$.

Formula (37) is encoded as a procedure `cmulWalsh3` in `CLIFFORD`, a Maple package for computations with Clifford algebras [2, 7]. It has the following pseudo-code.

```
1 cmulWalsh3:=proc(eI::clibasmon,eJ::clibasmon,B1::{matrix,list(nonnegint)})
2 local a,b,ab,monab,Bsig,flag,i,dim_V_loc,ploc,qloc,_BSIGNATUREloc;
3 global dim_V,_BSIGNATURE,p,q;
4 options 'Copyright (c) 2015-2016 by Rafal Ablamowicz and Bertfried Fauser.
5        All rights reserved.';
6 if type(B1,list) then
7   ploc,qloc:=op(B1);
8   dim_V_loc:=ploc+qloc:
9   _BSIGNATUREloc:=[ploc,qloc]:
10  else
11  ploc,qloc:=p,q; <<<-- this reads global p, q
12  dim_V_loc:=dim_V: <<<-- this reads global dim_V
13  _BSIGNATUREloc:=[ploc,qloc]:
14  if not _BSIGNATURE=[ploc,qloc] then _BSIGNATURE:=[p,q] end if:
15 end if:
16 a:=convert(eI,clibasmon_to_binarytuple,dim_V_loc);
17 b:=convert(eJ,clibasmon_to_binarytuple,dim_V_loc);
18 ab:=oplus(a,b);
19 monab:=convert(ab,binarytuple_to_clibasmon);
20 return twist(a,b,_BSIGNATUREloc)*Walsh(a,hinversegGrayCode(b))*monab;
21 end proc:
```

Now let us take a real quadratic vector space (V, Q) with a degenerate quadratic form Q such that $\dim V^{\perp} = r$, while Q restricted to the orthogonal complement of $V^{\perp}$ in V has signature (p, q), $(\dim V = n = p + q + r)$, and we let a basis $\mathbf{e}_i, 1 \leq i \leq n$ be such that $Q(\mathbf{e}_i) = 1$ resp. $Q(\mathbf{e}_i) = -1$, resp. $Q(\mathbf{e}_i) = 0$, for $0 \leq i \leq p$, resp. $p + 1 \leq i \leq p + q$, resp. $p + q + 1 \leq i \leq p + q + r$. We can now generate a universal Clifford algebra as the graded tensor product $C\ell_{p,q}r \cong C\ell_{p,q} \hat{\otimes} \bigwedge V^{\perp}$ with a Clifford product obtained by modifying the above formula (37) as follows: we introduce an extra scalar factor in front of $\mathbf{e}_{\underline{a}\oplus\underline{b}}$. This factor equals 1 or, resp. 0, depending whether the monomial elements $\mathbf{e}_{\underline{a}}$ and $\mathbf{e}_{\underline{b}}$ do not share, resp. do share, a common basis element $\mathbf{e}_i$ which squares to 0 in $C\ell_{p,q,r}$, that is, such that $Q(\mathbf{e}_i) = 0$.

A modified pseudo-code of such procedure called `cmulWalsh3pqr` has been encoded in a new experimental package `eClifford` for computations in $C\ell_{p,q,r}$ [1].

```
1 cmulWalsh3pqr:=proc(eI::eclibasmon,eJ::eclibasmon,B1::list(nonnegint))
2 local ploc,qloc,rloc,dim_V_loc,_BSIGNATUREloc,a,b,ab,monab,maxmaxindex,r_factor;
3 global twist,Walsh,hinverseGrayCode,oplus;
4 options 'Copyright (c) 2015-2016 by Rafal Ablamowicz and Bertfried Fauser.
5        All rights reserved.';
6 if nops(B1)=2 then
7   ploc,qloc:=op(B1);
8   rloc:=0;
9 elif nops(B1)=3 then
10  ploc,qloc,rloc:=op(B1);
11 else
12   error 'three non-negative integers p,q,r are needed in the list entered as
13        the last argument but received \%1 instead',B1
14 end if;
15 dim_V_loc:=ploc+qloc+rloc:
16 maxmaxindex:=max(op(eClifford:-eextract(eI)),op(eClifford:-eextract(eJ)));
17 if evalb(maxmaxindex>dim_V_loc) then
18   error 'maximum index \%1 found in the arguments \%2 and \%3 is larger
19        then dim_V = \%4 derived from the last argument \%5',
20        maxmaxindex,eI,eJ,dim_V_loc,B1
21 end if;
22 _BSIGNATUREloc:=[ploc,qloc]:
```

```
23 a:=convert(eI,eclibasmon_to_binarytuple,dim_V_loc);
24 b:=convert(eJ,eclibasmon_to_binarytuple,dim_V_loc);
25 if rloc=0 then
26    r_factor:=1
27 else
28    r_factor:=mul((1+(-1)^(a[i]*b[i]))/2,i=ploc+qloc+1..(ploc+qloc+rloc));
29 end if;
30 if r_factor=0 then return 0 else
31    ab:=oplus(a,b);
32    monab:=convert(ab,binarytuple_to_eclibasmon);
33    return twist(a,b,_BSIGNATUREloc)*Walsh(a,hinversegGrayCode(b))*monab;
34 end if;
35 end proc:
```

In the above, the code lines 25-33 accommodate the additional factor called `r_factor` which equals 1 or 0 as indicated above[5]. In particular, the Clifford algebra $Cl_{0,0,n} \cong \bigwedge V$, the exterior (Grassmann) algebra.

6. CLIFFORD ALGEBRAS $Cl_{p,q}$ AS TWISTED GROUP ALGEBRAS

In this last section we give a formal definition of a *twisted group ring* (algebra) following Passman [26, Section 2], and briefly refer to the paper by Albuquerque and Majid [25] in which the authors discuss twisting of a real group algebra of $(\mathbb{Z}_2)^n$ by using Hopf algebraic methods.

Definition 13 (Passman [26]). The *twisted group ring* $k^t[G]$ [26, Sect. 2] is an associative k-algebra, k is a field, with a basis $\{\bar{x} \mid x \in G\}$ and multiplication defined distributively for all $x, y \in G$ as

$$(38) \qquad \bar{x}\bar{y} = \gamma(x,y)\,\overline{xy}, \qquad \gamma(x,y) \in k^\times = k \setminus \{0\}.$$

where the function $\gamma : G \times G \to k^\times$ satisfies

$$(39) \qquad \gamma(x,y)\gamma(xy,z) = \gamma(y,z)\gamma(x,yz), \quad \forall z, y, z \in (\mathbb{Z}_2)^n \quad \text{(cocycle condition)}$$

to assure associativity $(\bar{x}\bar{y})\,\bar{z} = \bar{x}\,(\bar{y}\,\bar{z})$ in $k^t[G]$ for any $x, y, z \in G$.

Lemma 3 (Passman [26]). *The following relations hold in $k^t[G]$.*

(i) $\gamma(1,1)^{-1}\bar{1}$ *is the identity in $k^t[G]$;*
(ii) $\bar{x}^{-1} = \gamma(x,x^{-1})\gamma(1,1)^{-1}\overline{x^{-1}} = \gamma(x^{-1},x)\gamma(1,1)^{-1}\overline{x^{-1}}, \forall x \in G$;
(iii) $(\bar{x}\bar{y})^{-1} = \bar{y}^{-1}\bar{x}^{-1}, \forall x, y \in G$.

If $\gamma(1,1) = 1$ in part (i) of the above lemma, then we call γ *normalized*, which can always be achieved by scaling. In part (ii), the inverse $\bar{x}^{-1}$ is the result of the action of the antipode on $\bar{x}$ in the Hopf algebra sense, or, it can be viewed as the (un-normalized) action of the transposition map $T_{\tilde{\varepsilon}}$ introduced in [3–5] and mentioned in Section 3.1.

For a Hopf algebraic discussion of Clifford algebras $Cl_{p,q}$ as twisted group algebras $\mathbb{R}^t[(\mathbb{Z}_2)^n]$, where the twisting is accomplished via a 2-cocycle F which twists the group algebra $k[(\mathbb{Z}_2)^n]$ into a cotriangular Hopf algebra with a suitable cotriangular structure $\mathcal{R}$, see [8, 16] and references therein. Note that if γ is trivial, then the twist is trivial and the twisted group algebra is just the group algebra $k[G]$; if it is given by the XOR function on binary tuples, we get the Grassmann product (including a graded tensor product, or a graded switch; if γ is the choice described by Lounesto in (37), we get the Clifford algebra $Cl_{p,q}$ [9].

---

[5]Note that such factor can also be computed by an XOR operation [9].

7. Conclusions

As stated in the Introduction, the main goal of this survey paper has been to collect and summarize properties of certain finite 2-groups which appear in Clifford algebras $Cl_{p,q}$. On one hand, these Salingaros-defined groups $G_{p,q}$ appear as subgroups of the group of invertible elements. These subgroups play an important role in relation to the set of orthogonal primitive idempotents, with the help of which one defines spinorial representations. It has been observed by Salingaros, that these groups belong to five non-isomorphic families. On the other hand, one knows that all Clifford algebras $Cl_{p,q}$ are classified into five different families of simple and semisimple algebras depending on the values of (p, q) and $p + q$ (the Periodicity of Eight). Another connection with finite Salingaros groups appears via Chernov's observation that the algebras $Cl_{p,q}$ can be viewed as images of group algebras, most likely of the groups $G_{p,q}$ modulo a suitable ideal generated by a central nontrivial idempotent in the group algebra. This shows that the theory of extra-special 2-groups has a direct bearing on the structure of the Clifford algebras $Cl_{p,q}$. Finally, we have observed how Clifford algebras can be obtained by twisting a group algebra of $(\mathbb{Z}_2)^n$, either by using the Walsh functions, or equivalently but in a more sound mathematical way, by using a 2-cocycle and the formalism of cotriangular Hopf algebras [16].

8. Acknowledgments

Appendix A. The Structure Theorem on Clifford Algebras

In this appendix we list the main structure theorem for real Clifford algebras $Cl_{p,q}$. For more information on Clifford algebras, see [14, 20, 21].

Structure Theorem. *Let $Cl_{p,q}$ be the universal Clifford algebra over (V, Q), Q is non-degenerate of signature (p, q).*

(a) *When $p - q \neq 1$ mod 4 then $Cl_{p,q}$ is a simple algebra of dimension 2^{p+q} isomorphic with a full matrix algebra $\mathrm{Mat}(2^k, \mathbb{K})$ over a division ring $\mathbb{K}$ where $k = q - r_{q-p}$ and r_i is the Radon-Hurwitz number.[6] Here $\mathbb{K}$ is one of $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ when $(p - q)$ mod 8 is 0, 2, or 3, 7, or 4, 6.*

(b) *When $p - q = 1$ mod 4 then $Cl_{p,q}$ is a semisimple algebra of dimension 2^{p+q} isomorphic to $\mathrm{Mat}(2^{k-1}, \mathbb{K}) \oplus \mathrm{Mat}(2^{k-1}, \mathbb{K})$, $k = q - r_{q-p}$, and $\mathbb{K}$ is isomorphic to $\mathbb{R}$ or $\mathbb{H}$ depending whether $(p - q)$ mod 8 is 1 or 5. Each of the two simple direct components of $Cl_{p,q}$ is projected out by one of the two central idempotents $\frac{1}{2}(1 \pm \mathbf{e}_{12...n})$.*

(c) *Any element f in $Cl_{p,q}$ expressible as a product*

$$(40) \qquad f = \frac{1}{2}(1 \pm \mathbf{e}_{\underline{i}_1})\frac{1}{2}(1 \pm \mathbf{e}_{\underline{i}_2})\cdots\frac{1}{2}(1 \pm \mathbf{e}_{\underline{i}_k})$$

where $\mathbf{e}_{\underline{i}_j}$, $j = 1, \ldots, k$, are commuting basis monomials in $\mathcal{B}$ with square 1 and $k = q - r_{q-p}$ generating a group of order 2^k, is a primitive idempotent in $Cl_{p,q}$.

---

[6]The Radon-Hurwitz number is defined by recursion as $r_{i+8} = r_i + 4$ and these initial values: $r_0 = 0$, $r_1 = 1$, $r_2 = r_3 = 2$, $r_4 = r_5 = r_6 = r_7 = 3$.

Furthermore, $C\ell_{p,q}$ has a complete set of 2^k such primitive mutually annihilating idempotents which add up to the unity 1 of $C\ell_{p,q}$.

(d) *When $(p-q) \bmod 8$ is $0, 1, 2,$ or $3, 7,$ or $4, 5, 6$, then the division ring $\mathbb{K} = fC\ell_{p,q}f$ is isomorphic to $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$, and the map $S \times \mathbb{K} \to S,\ (\psi, \lambda) \mapsto \psi\lambda$ defines a right $\mathbb{K}$-module structure on the minimal left ideal $S = C\ell_{p,q}f$.*

(e) *When $C\ell_{p,q}$ is simple, then the map*

$$(41) \qquad C\ell_{p,q} \xrightarrow{\ \gamma\ } \mathrm{End}_{\mathbb{K}}(S), \quad u \mapsto \gamma(u), \quad \gamma(u)\psi = u\psi$$

gives an irreducible and faithful representation of $C\ell_{p,q}$ in S.

(f) *When $C\ell_{p,q}$ is semisimple, then the map*

$$(42) \qquad C\ell_{p,q} \xrightarrow{\ \gamma\ } \mathrm{End}_{\mathbb{K} \oplus \hat{\mathbb{K}}}(S \oplus \hat{S}), \quad u \mapsto \gamma(u), \quad \gamma(u)\psi = u\psi$$

gives a faithful but reducible representation of $C\ell_{p,q}$ in the double spinor space $S \oplus \hat{S}$ where $S = \{uf \,|\, u \in C\ell_{p,q}\}$, $\hat{S} = \{u\hat{f} \,|\, u \in C\ell_{p,q}\}$ and $\hat{\ }$ stands for the grade-involution in $C\ell_{p,q}$. In this case, the ideal $S \oplus \hat{S}$ is a right $\mathbb{K} \oplus \hat{\mathbb{K}}$-module structure, $\hat{\mathbb{K}} = \{\hat{\lambda} \,|\, \lambda \in \mathbb{K}\}$, and $\mathbb{K} \oplus \hat{\mathbb{K}}$ is isomorphic to $\mathbb{R} \oplus \mathbb{R}$ when $p - q = 1 \bmod 8$ or to $\mathbb{H} \oplus \hat{\mathbb{H}}$ when $p - q = 5 \bmod 8$.

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