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PROOF FRAMEWORKS - A WAY TO GET STARTED

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PROOF FRAMEWORKS—A WAY TO GET STARTED

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Abstract: *Many mathematics departments have instituted transition-to-proof courses for second semester sophomores to help them learn how to construct proofs and prepare for proof-based courses, such as abstract algebra and real analysis. We have developed a way of getting students, who often stare at a blank piece of paper not knowing what to do, started on writing proofs. This is the technique of writing proof frameworks, based on the logical structure of the statement of the theorem and associated definitions. Often there is both a first-level and a second-level proof framework.*

Key Words: *Transition-to-proof courses, proof frameworks, operable interpretations of definitions.*

1. INTRODUCTION

Have you ever heard a student say about writing proofs, "I just don't know where to start"? This is a commonplace lament heard by just about every transition-to-proof course teacher. It has even been documented several times in the mathematics education research literature. For example, Moore [7], in one of the first studies of transition-to-proof courses, found seven major sources of students' difficulties in constructing proofs, one of them being,

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"The students did not know how to use definitions to obtain the overall structure of proofs . . . despite diligent studying." Baker and Campbell [1] investigated what hinders transition-to-proof course students' proof writing. They observed that "While it appeared that the students seemed to have little trouble understanding and comprehending the basic laws of logic … many students experienced a type of mental block when the transition was made into mathematical proof construction, seeming either unable to comprehend the purpose of this type of writing or unclear on how to proceed in constructing such arguments." Edwards and Ward [4], in their research on students' (mis)use of definitions, found that, "Many students do not use definitions the way mathematicians do, *even in the apparent absence of any other course of action*." Selden, McKee, and Selden [10] reported instances of students' tendencies to write proofs from the "top-down" and their reluctance to unpack and use the conclusion to structure their proofs.

Students' difficulty getting started on writing proofs is not in question—those of us who teach transition-to-proof courses have seen it. The question is: What can be done about it? We think we have an effective solution to the difficulty; namely, begin by teaching students to write *proof frameworks*[2]. Before going into detail about this technique, we would like to tell you a little about how we teach our transition-to-proof courses.

2. OUR TRANSITION-TO-PROOF COURSES

[2] Selden and Selden first introduced the idea of proof frameworks in [15], but have expanded on it since then.

We feel that students learn to construct proofs best by learning through experience—that learning to prove theorems is a skill, somewhat like learning a physical skill. Thus, when we teach a transition-to-proof course, whether for undergraduates or for beginning graduate students in need of help in writing proofs, we maximize student experiences by teaching from brief notes and not lecturing. We have students present their proofs in class[3] and introduce them to a way of getting started writing proofs that we call *proof frameworks*. The course notes contain definitions, requests for examples, questions, and statements of theorems to prove. The students prove the theorems outside of class, present their proof attempts in class on the blackboard, and receive extensive critiques. The critiques consist of careful, oral, line-by-line readings to determine if they guarantee the theorem is true, along with occasional suggestions. This is followed by a second reading of the students' proof attempts, indicating how they might have been written in "better style" to conform to the genre of proofs [12]. Once these corrections and suggestions have been made, the student who made the proof attempt is asked to write it up carefully, incorporating any corrections and suggestions, for duplication for the entire class. In this way, by the end of the semester, all students receive a correct, well-written proof for each theorem in the course notes. Sometimes, if the students seem to need it, there are mini-lectures on topics such as logic or proof by contradiction.

[3] In some ways, the course resembles a very Modified Moore Method course [3, 5] or a science lab.

The homework, assigned each class period, consists of requests for proofs of the next two or three theorems in the course notes. It is not graded, but at the beginning of the next class, students show their proof attempts to the first author, who determines "on the spot", based on the students' written work, which students will be asked to present their proof attempts on the blackboard that day. Occasionally, a proof attempt with an interesting error is selected over other proofs. In addition to presenting their attempted proofs in class, the students have both take-home and in-class final examinations, each of which consists of four theorems, new to them, to prove. The mathematical topics considered in the course include sets, functions, continuity, and beginning abstract algebra in the form of a few theorems about semigroups and homomorphisms, and for graduate students, some point set topology. The teaching aim is to facilitate students' learning of the proof construction *process* through experience by having them construct as many different kinds of proofs as possible[4], especially in abstract algebra and real analysis, and *not* to learn a particular mathematical content.

We mention several times to our classes that, although we occasionally note especially good work, we are *not* grading them on the theorems they present and that we regard mistakes as opportunities to learn. Most important, we point out that telling another student how to prove a theorem is not a favor, as it takes away the other student's opportunity to

[4] In large classes, while students may not get to the board very often, they benefit from seeing other students' proofs critiqued.

experience how to prove it.

Next we discuss students' difficulty with where to start writing proofs.

3. PROOF FRAMEWORKS AND ACTIONS IN PROVING

When students are first learning to prove theorems, many actions[5] can become routine. It may seem surprising, but a great deal of what one does when writing a straightforward proof can be done almost on "automatic pilot", rather like stopping a car at a red light while talking to a passenger. One of the things that can be done this way is the writing of a proof framework to structure a straightforward proof. So what is a proof framework?

Proof frameworks are determined by the logical structure of the statement of the theorem to be proved. The most common form of theorem in our course notes is: Some quantified variables; then "If P", where P is a predicate about those variables; followed by "then Q", where Q is another predicate involving some of the variables.

A proof framework starts by introducing the variables. If, for example, "For all $a \in A$" occurs in the theorem statement, one writes in the emerging proof "Let $a \in A$", in which case a is henceforth regarded as fixed, but unspecified. If "there is a $b \in B$" occurs, one must "find/create" such a b and leave a space to insert or explain that. Typically, one works out what b should be near the end of a proof construction. If the quantifiers are mixed, some "for all" and some "there exist", then in the proof these should be introduced in the

[5] Our theoretical perspective on actions in the proving process can be found in Selden and Selden [13].

same order as in the statement of the theorem. This avoids inadvertently changing the meaning of the theorem during the proving process[6]. Where the theorem statement says "If *P*", one writes in the proof, "Suppose *P*" and leaves a space for further portions of the proof. Where the theorem statement says "then *Q*", one writes at the end of the emerging proof "Therefore *Q*". This produces the *first-level* of the proof framework.

At this point, in the process of writing a proof framework, the student should focus on *Q* and "unpack" its meaning, that is, remember or look up its definition, being careful to change the names of the variables to fit the proof at hand. It may happen that the meaning of *Q* has the same logical form as the original theorem. In that case, one can repeat the above process, providing a *second-level* proof framework which is written into the blank space between the "Suppose *P* and the "Therefore *Q*". If in writing the second-level framework, some variables have already been introduced, one does not re-introduce them.

All of this may seem rather complicated to explain, but it is much easier to understand in practice, and is illustrated below in our sample proofs. Also, once students can produce and use a proof framework for the above "If P, then Q" logical structure, it appears to be relatively easy to introduce frameworks for the seven or so other logical structures in our course[7]. Finally, we are not

[6] The need for this is easily seen by comparing "For all $x \in \mathbb{N}$, there is a $y \in \mathbb{N}$ such that $x \leq y$" with "There is a $y \in \mathbb{N}$ such that for all $x \in \mathbb{N}$, $x \leq y$".
[7] For example, a proof involving cases or a proof by contradiction each have their own proof framework.

claiming that mathematicians write proofs in the way we are describing, but only that doing so will be helpful for students and that mathematicians will accept the results.

Next we will provide a sample proof framework of an elementary set theory theorem.

4. A PROOF FRAMEWORK FOR A THEOREM ABOUT SETS

Here is the theorem:

Theorem. If A, B, and C are sets and $C\backslash B \subseteq C\backslash A$, then $C \cap A \subseteq C \cap B$.

We will illustrate our technique by writing the first-level proof framework, namely, using the "If" part to write the first line of the emerging proof, skipping a space (here indicated by an ellipsis), and then writing the last line of the proof. Thus:

Theorem. If A, B, and C are sets and $C\backslash B \subseteq C\backslash A$, then $C \cap A \subseteq C \cap B$.

Proof:

Let A, B, and C be sets. Suppose $C\backslash B \subseteq C\backslash A$.

• • •

Therefore, $C \cap A \subseteq C \cap B$. QED.

This completes the first-level framework. Next, one should focus on the last line of the proof to determine what is to be proved. In this case, it is a set inclusion. One then looks up, or remembers, the definition of set inclusion and uses it in its operable form (discussed below). Accordingly, one knows how the second line and the second-to-last line should be written. Thus:

Theorem. If A, B, and C are sets and $C \backslash B \subseteq C \backslash A$, then $C \cap A \subseteq C \cap B$.

Proof:

Let A, B, and C be sets. Suppose $C \backslash B \subseteq C \backslash A$.

Suppose $x \in C \cap A$.

• • •

Thus $x \in C \cap B$.

Therefore, $C \cap A \subseteq C \cap B$. QED.

This gets one a long way towards a proof, but of course, there next needs to be some genuine problem solving, and perhaps exploration on scratch work, to fill in the middle (the ellipsis). For completeness, we give the entire proof. Thus:

Theorem. If A, B, and C are sets and $C \backslash B \subseteq C \backslash A$, then $C \cap A \subseteq C \cap B$.

Proof:

Let A, B, and C be sets. Suppose $C \backslash B \subseteq C \backslash A$.

Suppose $x \in C \cap A$. So $x \in C$ and $x \in A$.

Suppose to the contrary $x \notin B$.

Then $x \in C \backslash B$, so $x \in C \backslash A$. Thus $x \notin A$. This is a contradiction.

So $x \in B$.

Thus $x \in C \cap B$.

Therefore, $C \cap A \subseteq C \cap B$. QED.

We are not claiming that teaching mid-level university mathematics students to write proof frameworks is a panacea, but it gives students a start so they won't just stare at a blank piece of paper.

To indicate the usefulness of proof frameworks, we provide a real life anecdote. We taught a graduate student who, considerably later, was taking her real analysis PhD comprehensive examination consisting of eight (original to her) theorems to prove in three hours. Afterwards, she reported to us that upon reading one of the theorems, she panicked, but then controlled herself and decided to write a proof framework. Just this simple act oriented her to the "real problem" at hand, relieved her anxiety, and allowed her to proceed to construct a proof.

5. A PROOF FRAMEWORK FOR A REAL ANALYSIS THEOREM

Writing proof frameworks can get more complicated as the proofs themselves get more complicated and as they begin to involve several mixed quantifiers. For beginning students of real analysis, proofs of the ε–N or ε–δ type are notoriously difficult, both to understand and to structure. We will give an example using the following theorem on sequence convergence[8]:

Theorem: Let $\{a_n\}$ and $\{b_n\}$ be sequences, both converging to L. If $\{c_n\}$ is the sequence given by $c_n=a_n$ when n is even and $c_n=b_n$ when n is odd, then $\{c_n\}$ converges to L.

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[8] This theorem was previously used in McKee, Savic, Selden, and Selden [6] to illustrate proof co-construction in a real analysis proving supplement.

As with the above theorem about sets, one begins by writing a first-level framework. To keep track of the order of the proving actions, we will number them sequentially using square brackets (e.g., [1]).

Theorem: Let $\{a_n\}$ and $\{b_n\}$ be sequences, both converging to L. If $\{c_n\}$ is the sequence given by $c_n=a_n$ when n is even and $c_n=b_n$ when n is odd, then $\{c_n\}$ converges to L.

Proof: [1] Let $\{a_n\}$ and $\{b_n\}$ be sequences and L be a number so that $\{a_n\}$ and $\{b_n\}$ converge to L. Suppose $\{c_n\}$ is the sequence given by $c_n=a_n$ when n is even and $c_n=b_n$ when n is odd.

• • •

[2] Therefore $\{c_n\}$ converges to L. QED.

This completes the first-level framework. Next, to write the second-level proof framework, one unpacks the definition of sequence convergence and writes [3], [4]. [5], and [6], leaving three blank spaces, designated below by ellipses, to be filled in after one has done some scratch work calculations. Thus:

Theorem: Let $\{a_n\}$ and $\{b_n\}$ be sequences, both converging to L. If $\{c_n\}$ is the sequence given by $c_n=a_n$ when n is even and $c_n=b_n$ when n is odd, then $\{c_n\}$ converges to L.

Proof: [1] Let $\{a_n\}$ and $\{b_n\}$ be sequences and $L\in\mathbb{R}$ so that $\{a_n\}$ and $\{b_n\}$ converge to L. Suppose $\{c_n\}$ is the sequence given by $c_n=a_n$ when n is even and $c_n=b_n$ when n is odd.

[3] Let $\varepsilon \in \mathbb{R}$ and $\varepsilon > 0$.

. . .

[4] Let $N \in \mathbb{N}$ and N = ...

[5] Suppose $n > N$.

. . .

[6] Then $| c_n - L| < \varepsilon$.

[2] Therefore $\{c_n\}$ converges to L.   QED.

This gets one a long way towards a proof, but of course, one needs to use the hypotheses ([7] and [8] below), "play with" the absolute value condition on ε, notice the joint dependency of N_a and N_b on ε, and conjecture an appropriate value for *N*. Doing so involves some genuine problem solving, and perhaps exploration in scratch work, to fill in the three spaces (the ellipses). For completeness, we next give the entire proof:

Theorem: Let $\{a_n\}$ and $\{b_n\}$ be sequences, both converging to L. If $\{c_n\}$ is the sequence given by $c_n = a_n$ when n is even and $c_n = b_n$ when n is odd, then $\{c_n\}$ converges to L.

Proof: [1] Let $\{a_n\}$ and $\{b_n\}$ be sequences and L be a number so that $\{a_n\}$ and $\{b_n\}$ converge to L. Suppose $\{c_n\}$ is the sequence given by $c_n = a_n$ when n is even and $c_n = b_n$ when n is odd.

[3]] Let $\varepsilon \in \mathbb{R}$ and $\varepsilon > 0$.

[7] Since $\{a_n\}$ converges there exists an $N_a \in \mathbb{N}$ such that for all $i \in \mathbb{N}$ if $i > N_a$, then $|a_i - L| < \varepsilon$.

[8] Since $\{b_n\}$ converges there exists an $N_b \in \mathbb{N}$ such that for all $j \in \mathbb{N}$ if $j > N_b$,

$|b_j - L| < \varepsilon$.

[4] Let $N \in \mathbb{N}$ and

[9] $N = \max\{N_a, N_b\}$.

[5] Suppose $n > N$.

[10] Case 1: Suppose n is even. Then $|c_n - L| = |a_n - L| < \varepsilon$.

[11] Case 2: Suppose n is odd. Then $|c_n - L| = |b_n - L| < \varepsilon$.

[6] Then $|c_n - L| < \varepsilon$.

[2] Therefore $\{c_n\}$ converges to L. QED.

Students do not always come up with $N=\max\{N_a, N_b\}$. Sometimes they come up with $N=N_a+N_b$, which also works. It is clear that having students learn to construct proof frameworks is indeed just a way to get started and there is really a great deal more to proving that cannot be proceduralized. But we are only claiming that the writing of proof frameworks gets students started. Once they get started and begin to write straightforward proofs, they often get a feeling that writing proofs is something they really *can* do. Then a budding sense of self-efficacy, along with persistence[9] will enable them to continue the journey to writing ever harder proofs.

For the above ε–N proof, the order of the sequence of proving actions is quite different from the top-down order of the proof text itself. A student who had

[9] Much of our theoretical perspective on the role of self-efficacy and persistence can be found in [11].

not experienced this technique, who was trying to prove a similar but more difficult, theorem, might come up with a collection of relevant, but unordered pieces of information, which might then have to be assembled, rather like a jigsaw puzzle. We believe the need for such assembly would add to the difficulty of constructing this kind of proof.

Unfortunately, early on there is another student difficulty that can impede the writing of a proof, and in particular, a second-level proof framework -- how to use a mathematical definition to structure a proof. Students need to be able to "unpack" mathematical definitions, as was done above in the sample set theory and sequence convergence proofs, in order to write a second-level proof framework.

6. UNPACKING DEFINITIONS TO WRITE A SECOND-LEVEL PROOF FRAMEWORK

In addition to knowing how to get started, that is, to write at least a first-level proof framework, we have noticed another problem with university students just beginning to prove theorems. Just as observed by Moore [7] and Edwards and Ward [4], beginning university students don't consider mathematical definitions in the same way mathematicians do. Furthermore, they often don't know how to use them as mathematicians would, even when they are considering what we would call straightforward definitions. Thus, we have come up with the idea of operable interpretations[10] of definitions.

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[10] This is similar to Bills and Tall's [2] idea of operable definitions.

Consider the following definition of the image of a set A under a function f. Let f: $X \to Y$ be a function and $A \subseteq X$. Then $f(A) = \{ y \mid y \in Y$ and there is $a \in A$ so that $f(a) = y\}$. To use this definition, one really needs the following *operable interpretation*. That is, one needs to know that if one has "$q \in f(A)$", one can say "there is $p \in A$ so that $f(p) = q$". And conversely, one also needs to know that if one knows "$p \in A$", then one can say "$f(p) \in f(A)$". One might expect that beginning transition-to-proof course students would be able to come to such operable interpretations of definitions by themselves, but we have observed that many do not.

To help our undergraduate transition-to-proof course students with learning to use mathematical definitions in their operable versions, we have provided students with two-column handouts containing operable interpretations of definitions that are in our notes. In the left-hand column is the definition as stated in our notes. Across from it, in the right-hand column, is the corresponding operable interpretation. That is, given the definition of the image of a set A under a function f, we would write across from the mathematical definition, "If you know $q \in f(A)$, you can say 'there is $p \in A$ so that $f(p) = q$'. Also, if you know $p \in A$, then one can say '$f(p) \in f(A)$'". Since for this course we are interested in getting students to construct proofs and not especially in having them memorize definitions, theorems, or their proofs, we let students use both the course notes and these operable interpretation handouts not only on homework, but also on both the mid-term and final examinations.

7. EXAMPLE OF NOT UNPACKING THE CONCLUSION AND WRITING FROM THE TOP-DOWN

We have observed that, although some students may have misconceptions that impede their ability to write proofs, what actually seems to stop many students is beneficial actions they *do not* take, such as writing a proof framework, as well as detrimental actions they *do* take [9, 10]. Indeed, students often want to write proofs from the "top-down". But, this detrimental action can divert students away from unpacking the conclusion and focusing on what is to be proved.

To illustrate, we describe the behavior of Willy[11], a student in our graduate "proofs course". Willy was trying to prove, toward the end of the course, the following: Theorem: *Let X and Y be topological spaces and $f: X \to Y$ be a homeomorphism of X onto Y. If X is a Hausdorff space, then so is Y.* Willy had indicated that he had not yet proved the theorem. Thus, because only a little class time remained, we asked Willy to go to the board and write a proof framework. We thought this would be an easy task because, early on in the course, Willy had developed some ability to write proof frameworks.

On the left side of the board, Willy wrote:

Proof. Let X and Y be topological spaces.

[11] Willy's behavior was described previously in Selden, McKee, and Selden [10] and we have adapted that description here.

Let $f : X \to Y$ be a homeomorphism of X onto Y.

Suppose X is a Hausdorff space.

. . .

Then Y is a Hausdorff space.

Then, on the right side of the board, he listed:

homeomorphism

one-to-one

onto

continuous (f is open mapping)

and then looked perplexedly back at the left side of the board. Even after two hints to look at the final line of his proof, Willy said, "And, I was just trying to just think, homeomorphism means one-to-one, onto, …" After some discussion about the meaning of homeomorphism, the first author said, "There is no harm in analysing what stuff you might want to use, but there is more to do before you can use any of that stuff", *meaning that the conclusion should be examined and unpacked first*. The final quoted sentence was an implicit expectation of what we hoped Willy would eventually do. While it would have been convenient to tell Willy this explicitly, we did not. This is because our expectation is that all students will learn to work autonomously, and we thought that Willy, given time, could figure out what to do. Willy did not make further progress that day.

We inferred that Willy was focusing on the meaning and potential uses of the hypotheses before examining and unpacking the conclusion. This behaviour can often lead to difficulties, as it did for Willy. We conjectured that Willy and other students, who are reluctant to look at and unpack the conclusion feel uncomfortable about this, or perhaps feel it more appropriate to begin with the hypotheses and work forward.

We also conjectured that, had Willy not been distracted by focusing on the meaning of homeomorphism, he might have written the second-level proof framework. That is, he might have filled in the blank space of his proof with something like:

Let y_1 and y_2 be two elements of Y.

. . .

Thus there are disjoint open sets U and V contained in Y so that $y_1 \in U$ and $y_2 \in V$.

As in Willy's case, writing the first- and second-level proof frameworks exposes the "real problem" to be solved to complete the proof. Indeed, by the next class meeting, Willy had constructed a proof in the way we had expected. Of course, in addition to writing the first- and second-level proof frameworks, Willy needed to invoke some conceptual knowledge about homeomorphisms and the Hausdorff property.

One might ask where the tendency of students to write proofs from the top-down comes from. According to Nachlieli and Herbst [8], it is the norm among U.S. high school geometry teachers to require students, when doing two-column geometry proofs, to follow every statement immediately by a reason, and hence write from the top-down. However, given sufficient experience with writing proof frameworks, the initial tendency of many university students to write proofs in a top-down fashion tends to fade. In addition, we think that automating the actions required to write a proof framework eventually resolves the top-down difficulty and exposes the “real problem” to be solved in order to complete the proof [14]. Of course, the students have to remain attentive to the course long enough for this to happen.

8. TEACHING IMPLICATONS

Because students, when faced with a theorem to prove, often look at a blank piece of paper and don’t know where to start, we suggest that perhaps a good place to start is with explicitly teaching proof frameworks. Indeed, one can spend the first day or two of class, whether a regular lecture class or an inquiry-based class like ours, just having students write first- and second-level proof frameworks for proofs of theorems, without having to complete their proofs.

Unfortunately, we have found that at first students tend to resist writing the second-level proof framework. We think this is because it involves writing in a way that is not from the “top-down”. We conjecture that, in most of their past experience, students have read proofs, and perhaps written them, from the top-down. We suggest there should be enough practice exercises for students, not only to understand the rationale for writing proof frameworks, but also to form the habit of consistently writing proof frameworks to structure proofs. That is, they should overcome their feeling of reluctance to write second-level proof frameworks. To accomplish this in a reasonable amount of time, it would probably be better to ask students to practice constructing only proof frameworks, not entire proofs, for practice problems. Early on, one might also consider giving very brief short-answer quizzes asking students to write both first- and second-level proof frameworks.

But there is the other difficulty, mentioned above, that occurs fairly often, namely, that of not using mathematical definitions correctly. Here we suggest that it might be a good idea to explicitly teach operable interpretations of definitions, perhaps by using our idea of two-column handouts. Alternatively, one might consider occasional brief small group discussions whose purpose is to develop operable interpretations for recently introduced definitions. At the end of such group discussions, the teacher might certify which interpretations are acceptable to the mathematical community. One might also consider very brief short-answer quizzes on operable interpretations of definitions.

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