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HOMEOMORPHISMS ON A CANTOR SET WITH SUBSEQUENTIALLY DENSE ORBITS

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ABSTRACT. We study homeomorphisms on a Cantor set K such that for any strictly increasing sequence $\{n_i\}_{i=1}^{\infty}$ of positive integers there exists a point $c \in K$ such that the sets $\mathcal{O}^+ = \{h^{n_i}(c), i = 1, 2, \ldots\}$ and $\mathcal{O}^- = \{h^{-n_i}(c), i = 1, 2, \ldots\}$ are both dense in K.

1. Introduction

Knaster stated the following problem: Let P and Q be nowhere dense and closed subsets of the Cantor set C and let f be a homeomorphism from P onto Q. Does there exist a homeomorphism h from C onto itself that is and extension of f? The first proof that this is the case, based on the notion of boolean algebras, was presented by C. Ryll-Nardzewski in 1951. A topological proof was published in [8]. In [3] it is shown that there exists an extension with the property that an orbit of some point is dense. Here we prove that there exists an extension h such that for every strictly increasing sequence $n_1, n_2, \ldots$ of positive integers there is a point $c \in C$ such that the sets $\{h^{n_i}(c) : i = 1, 2, \ldots\}$ and $\{h^{-n_i}(c) : i = 1, 2, \ldots\}$ are dense in C. Note that h^0 denotes an identity. We also show that a composition of a such a homeomorphism with itself is a homeomorphism with subsequentially dense orbits.

Definition 1.1. A homeomorphism h on a Cantor set C with the property that for some point $c \in C$ the orbit $\mathcal{O} = \{h^n(c) : n \text{ is an integer}\}$ is dense in C is called *transitive.*

Definition 1.2. A homeomorphism h on a Cantor set C has subsequentially dense orbits if for every strictly increasing sequence $n_1, n_2, \ldots$ of positive integers there is a point $c \in C$ such that the sets $\{h^{n_i}(c) : i = 1, 2, \ldots\}$ and $\{h^{-n_i}(c) : i = 1, 2, \ldots\}$ are dense in C.

2. Main Results

In [6] the following is prowen:

Theorem 2.1. *There is a homeomorphism h on a Cantor set K that has exactly two fixed points. Furthermore, for any increasing sequence $n_1, n_2, \ldots$*

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of positive integers there is a point $c \in K$ such that the sets $\{h^{n_i}(c) : i = 1, 2, \ldots\}$ and $\{h^{-n_i}(c) : i = 1, 2, \ldots\}$ are dense in K.

We show that in the above theorem a two point set can be replaced by any at most countable closed subset of a Cantor set.

Theorem 2.2. *For at most countable compact metric space F there is a Cantor set K and a homeomorphism h on K such that h is an identity on F and there are no other fixed points. Furthermore, for any increasing sequence $n_1, n_2, \ldots$ of positive integers there is a point $c \in K$ such that the sets $\{h^{n_i}(c) : i = 1, 2, \ldots\}$ and $\{h^{-n_i}(c) : i = 1, 2, \ldots\}$ are dense in K.*

Proof. The theorem has been proved in [6] for a two point set. If the set F has only one point then we can obtain the required result by identifying the two fixed points. So suppose that F has at least three points, say $F = \{a_\alpha : \alpha < \beta\}$, where β is at most countable ordinal.

We follow the proof of the the theorem 2.1 from [6].

Let Z be the set of all integers. Put $K = F^Z$. Let h be a homeomorphism from K onto itself such that a point $(t_i)_{i=-\infty}^{\infty}$ is a value of h at $(s_i)_{i=-\infty}^{\infty}$ if and only if $t_i = s_{i+1}$. We identify a point a_α in F with a point (a_i) in K with all coordinates equal a_α. It is clear that each point of F is a fixed point of h and h does not have other fixed points.

Let $\mathcal{A}$ be a family of all finite sequences of points of F. Let $\mathcal{B} = \mathcal{A} \times \mathcal{A}$. Write the family $\mathcal{B}$ as a sequence $\mathcal{B} = \{(r_k, s_k) : k = 1, 2, \ldots\}$. Let $l(r_k)$ and $l(s_k)$ be the lengths of the finite sequences r_k and s_k respectively. Let $n_1, n_2, \ldots$ be an increasing sequence of positive integers. We may assume that $n_1 > l(r_1) + l(s_1) + l(r_2) + l(s_2)$ and $n_{k+1} - n_k > l(r_k) + l(s_k) + l(r_{k+1}) + l(s_{k+1})$ for $k = 1, 2, \ldots$. Otherwise we may choose a subsequence satisfying those conditions and rename it.

Suppose that $r_k = (r_k^1, r_k^2 \ldots, r_k^{l(r_k)})$ and $s_k = (s_k^1, s_k^2 \ldots, s_k^{l(s_k)})$. Define a point c in K as follows:

$c_{n_k+i} = c_{-n_k+i} = r_k^{i+1}$, where $i = 0, 1, \ldots, l(r_k) - 1$ and $k = 1, 2, \ldots$,

$c_{n_k-i} = c_{-n_k-i} = s_k^i$, where $i = 1, 2, \ldots, l(s_k)$ and $k = 1, 2, \ldots$.

We put a_1 everywhere else.

Let U be a set in the standard basis of $\{0, 1\}^Z$. So U is determined by fixing elements of F in a finite number of places, say $j_1, j_2, \ldots, j_m$. where $j_1 < j_2 < \ldots < j_m$. So for any point u of U the coordinates $u_{j_1}, u_{j_2}, \ldots u_{j_m}$ are fixed. Let r be a finite sequence of length $j_m + 1$. So $r = (r^1, r^2, \ldots, r^{j_m+1})$ if $j_m \geq 0$. We put $r^i = u_{j_n+1}$ iff $i = j_n + 1$ and $j_n \geq 0$, and a_1 everywhere else. If $j_m < 0$ then we take r to be a sequence with only one element which is equal a_1. If $j_1 \geq 0$ we put s to be a sequence of length one having a_1 as the only value. If $j_1 < 0$ we take s to be a sequence of length $-j_1$, so $s = (s^1, s^2, \ldots, s^{-j_1})$. We put $s^i = u_{j_n}$ if $i = -j_n$ and $j_n < 0$, and a_1 everywhere else.

There is a positive integer k such that $r = r_k$ and $s = s_k$. So $h^{n_k}(c)$ and $h^{-n_k}(c)$ are points of U. Therefore the sets $\{h^{n_i}(c) : i = 1, 2, \ldots\}$ and $\{h^{-n_i}(c) : i = 1, 2, \ldots\}$ are dense in K. □

Remark 2.3. If a transitive homeomorphism on the Cantor set (that is a homeomorphism having a dense orbit) does not have any fixed points than it may not have subsequentially dense orbits. For example any transitive homeomorphism that can be used to construct a solenoid does not have subsequentially dense orbits.

Question 2.4. Suppose a homeomorphism h on a Cantor set K has at least one fixed point and the set of all fixed points is nowhere dense in K. If h is transitive does it have subsequentially dense orbits?

In what follows we need the following theorem from [8]:

Theorem 2.5. *Let P and Q be closed and nowhere dense subsets of the Cantor set C and let f be a homeomorphism from P onto Q. Then there exists a homeomorphism h from C onto itself which is an extension of f.*

Theorem 2.6. *Let P and Q be closed and nowhere dense subsets of the Cantor set C and let f be a homeomorphism from P onto Q. Then there exists an extension h of f such that h is a homeomorphism from C onto itself and has subsequentially dense orbits.*

Proof. We use an approach that van Mill used in a proof of *Theorem 3* in [5]. Note that in view of *Theorem* 2.5 we can assume that $P = Q$. Put $K = C^Z$, where Z is the set of all integers. We will write points of K in the form $(a_i)_{i=-\infty}^{\infty}$ or (a_i). W identify every point $p \in P$ with a point $(a_i) \in C^Z$ by putting $a_i = f^i(p)$. Let I_P denote this identification. Let s be a shift homeomorphism on C^Z, that is $s(a_i) = (b_i)$ where $b_i = a_{i+1}$. Then $s(I_P(p)) = I_P(f(p))$. Applying *Theorem* 2.5 we may get an extension of I_P to a homeomorphism f_P from C onto K.

Let D be a countable dense subset of C having no points in common with P, and therefore Q. As in the proof of the *theorem* 2.2 we put $\mathcal{A}$ to be a family of all finite sequences of points of D. Let $\mathcal{B} = \mathcal{A} \times \mathcal{A}$. Write the family $\mathcal{B}$ as a sequence $\mathcal{B} = \{(r_k, s_k) : k = 1, 2, \ldots\}$. Let $l(r_k)$ and $l(s_k)$ be the lengths of the finite sequences r_k and s_k respectively. Let $n_1, n_2, \ldots$ be an increasing sequence of positive integers. We may assume that $n_1 > l(r_1) + l(s_1) + l(r_2) + l(s_2)$ and $n_{k+1} - n_k > l(r_k) + l(s_k) + l(r_{k+1}) + l(s_{k+1})$ for $k = 1, 2, \ldots$. Otherwise we may choose a subsequence satisfying those conditions and rename it. We define a point $c \in K$ as in the proof of *theorem* 2.2. So the sets $\{s^{n_i}(c) : i = 1, 2, \ldots\}$ and $\{s^{-n_i}(c) : i = 1, 2, \ldots\}$ are dense in K.

Define a homeomorphism h from C onto itself by $h = f_P^{-1} \circ s \circ f_P$. Put $b = f_P^{-1}(c)$. Then $h^n(b) = (f_P^{-1} \circ s \circ f_P)^n(b) = f_P(s^n(c))$. So the sets $\{h^{n_i}(b) : i = 1, 2, \ldots\}$ and $\{h^{-n_i}(b) : i = 1, 2, \ldots\}$ are dense in K. □

Theorem 2.7. *Let h be a homeomorphism on a Cantor set K that has subsequentially dense orbits. Then for any integer n other than zero the homeomorphism h^n has subsequentially dense orbits.*

Proof. Let k be an integer other than zero. Let $n_1, n_2, \dots$ be a strictly increasing sequence of positive integers. Because h has subsequentially dense orbits, then for the sequence $|k| \cdot n_1, |k| \cdot n_2, \dots$ there exists a point $c \in K$ such that the sets $\{h^{|k| \cdot n_i}(c) : i = 1, 2, \dots\}$ and $\{h^{-|k| \cdot n_i}(c) : i = 1, 2, \dots\}$ are dense in K. But these sets are the same as $\{(h^k)^{n_i}(c) : i = 1, 2, \dots\}$ and $\{(h^k)^{-n_i}(c) : i = 1, 2, \dots\}$. □

So if h has subsequentially dense orbits, so does h^{-1}, however their composition is an identity. This brings the following question:

Question 2.8. Let h and g be homeomorphisms on a Cantor set K that have subsequentially dense orbits. Suppose that the composition $f \circ g$ is not an identity. Does it have subsequentially dense orbits?

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