DEPARTMENT OF MATHEMATICS

TECHNICAL REPORT

DIFFICULTIES FIRST-YEAR UNIVERSITY STUDENTS HAVE IN READING THEIR MATHEMATICS TEXTBOOK

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Difficulties First-year University Students Have in Reading Their Mathematics Textbooks

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Abstract
This exploratory study examined the experiences and difficulties certain first-year university students displayed in reading new passages from their mathematics textbooks. We interviewed eleven precalculus and calculus students who were considered to be good at mathematics, as indicated by high ACT mathematics scores. These students were also good general readers, as indicated by their high ACT reading comprehension scores and by their use of many of the metacognitive strategies developed in reading comprehension research. To gauge the effectiveness of students' reading of passages from their mathematics textbooks, we asked them to attempt straightforward mathematical tasks, based directly on what they had just read. Our students demonstrated enough difficulties with these tasks, that it appears they do not benefit from reading their textbooks as much as their teachers or textbook authors would hope. Analysis of the data suggests that the reading strategies used by these students were not sufficient for them to complete many of the tasks. Instruction or guidance in strategies that are specifically related to mathematics reading may be needed to help students deal with mathematical text.

Keywords: reading mathematics, first-year university students, precalculus, calculus

Introduction

From our own experience and in talking with colleagues, we have come to believe that many, perhaps most, first-year university students do not read large parts of their mathematics textbooks effectively. Whether this is because they cannot do so, or choose not to do so, seems not to have been established. However, there have been a number of calls for teachers to instruct students on how to read mathematics (Bratina & Lipkin, 2003; Cowen, 1991; Datta, 1993; DeLong & Winter, 2002; Draper, 2002; Fuentes, 1998; Pimm, 1987; Shuard & Rothery, 1988). Also the textbooks for many first-year university courses, such as college algebra, precalculus, and calculus seem to be written with the assumption that they will be read thoroughly and precisely. For example, this is suggested by the preface of the precalculus book used by our students:

> The following suggestions are made to help you get the most out of this book and your efforts. As you study the text we suggest a five-step approach. For each section,
>
> 1. Read the mathematical development.
> 2. Work through the illustrative examples.

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> 3. Work the matched problem.
> 4. Review the main ideas in the section
> 5. Work the assigned exercises at the end of the section.
>
> All of this should be done with a graphing utility, paper, and pencil at hand. In fact, no mathematics text should be read without pencil and paper in hand; mathematics is not a spectator sport. Just as you cannot learn to swim by watching someone else swim, you cannot learn mathematics by simply reading worked examples—you must work problems, lots of them. (Barnett, Ziegler, & Byleen, 2000).

In this exploratory study we examine whether first-year undergraduate mathematics students can read their mathematics textbooks effectively, that is, we examine (1) what they do when reading, (2) whether they benefit from their reading, and (3) what difficulties or obstacles they encounter.

In Section 1, we discuss how mathematics textbooks differ from other textbooks, describe the Constructively Responsive Reading framework (CRR), a theoretical framework developed in reading comprehension research (Pressley & Afflerbach, 1995), and point out some potential problems in reading mathematics. Next, in Section 2, we refine our research questions. In Section 3 we describe the students, their courses, and our research methodology; and in Section 4, we offer our observations concerning students' use of the reading strategies mentioned in the CRR framework and describe students' difficulties in working straightforward tasks. In Section 5, we discuss these observations and suggest some reading strategies from the CRR framework that teachers might stress. Finally, in Section 6, we suggest some implications for teaching and some suggestions for future research.

1. Background and Literature Review

1.1 Mathematical text

In mathematical writing, mathematicians appear to prize brevity, conciseness, and precision of meaning. Most first-year university mathematics textbooks currently published in the U.S. contain exposition, definitions, theorems and less formal mathematical assertions, graphs, figures, tables, examples,[4] and end of section exercises. Often the definitions, theorems, and examples are set apart from the expository text by boxes, colors, or spacing. Figures containing graphs and explanatory captions often appear in the margins. Typically there is a repeated pattern consisting of first presenting a bit of content, such as a definition or theorem and perhaps some less formal mathematical assertions, then a few closely related example tasks are worked out, and finally students are invited to work very similar tasks themselves. In these respects, the textbooks (Barnett, Ziegler, & Byleen, 2000; Larson, Hostetler, & Edwards, 2002) read by the students in this study appear to us to be typical. (See Appendices A and B.)

Some special features of mathematical text that can lead to student difficulties as indicated by Barton and Heidema (2002) and Shuard and Rothery (1988) include:

[4] The word "example," as used here, normally refers to mathematical tasks, some with solutions provided in the textbook, some without. This contrasts with some other mathematical writing where "example" refers to an object, such as 6 is an example of an even integer.

1. Reading mathematics often requires reading from right to left, top to bottom, bottom to top, or diagonally.
2. The text in mathematics textbooks has more concepts per sentence, per word, and per paragraph than ordinary textbooks.
3. Mathematical concepts are often abstract and require effort to visualize.
4. The text in mathematics textbook is terse and compact—that is, there is little redundancy to help readers uncover the meaning.
5. Words have precise meanings which often are not fully understood. Students' concept images[5] of them may be "thin," or stipulated meanings may be treated as extracted meanings.
6. Formal logic connects sentences so the ability to understand implications and make inferences across sentences is essential.
7. In addition to words, mathematics textbooks contains numeric and non-numeric symbols.
8. The layout of many mathematics textbooks can make it easy to find and read worked examples while skipping crucial explanatory text.
9. Mathematics textbooks often contains complex sentences which can be difficult to parse and understand.

In addition, definitions are to be read and used in an unusual way, and play an especially important role in mathematics. Readers of mathematical text must know how to read a definition as a stipulation of meaning, attending to every part with no extraneous connotations. Such definitions are unlike dictionary definitions which are often only approximate descriptions extracted from everyday language usage. Edwards and Ward (2004) indicated that even more advanced university students of mathematics have difficulty understanding the role and use of mathematical definitions. In our experience, even when students can correctly state and explain a mathematical definition, they may not use it correctly, because they do not understand the distinction between mathematical (stipulated) and dictionary (extracted) definitions.[6] For example, students may attend to only part of a mathematical definition or may inadvertently add some additional property. Precise use of language is also required in applying theorems, and we suspect leads to similar difficulties.

We also note that mathematics textbooks contain many complexes of symbols that function as ideographs rather than letters. The meaning of such complexes cannot be "spelled or sounded out" while students read, as is often the case with an unfamiliar word. The decoding of a word – from patterns of letters, to phonemes, to the sound of the word, to its meaning – occurs very largely outside of consciousness. In contrast, the

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[5] One's concept image (Tall and Vinner, 1981) is a mental construct including such knowledge as relevant examples, non-examples, facts, properties, relationships, diagrams, images, and visualizations, that one associates with the concept.

[6] In a stipulated, also called an analytic, definition one must use all parts of the definition and not infer additional conditions. Such a definition can bring a concept or mathematical entity into existence. In contrast an extracted, also called a synthetic or a dictionary, definition is a description of an already existing entity. One need not use all parts of such a definition and may appropriately infer additional conditions.

decoding and comprehension of a complex of mathematical symbols usually requires their conscious repetition in inner speech – and, probably considerable working memory. Consider

$$f(x) = 3x^2 - 5x + C$$

or

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad .$$

In converting these expressions to inner speech and comprehending them, students need to know that they represent implicitly universally quantified sentences, with "=" being the verb. That is, instead of a literal decoding ("*f*, left parenthesis, *x*, right parenthesis, equal sign, …"), the first example is read "*f* of *x* equals . . ." and the second is read from bottom to top and left to right.

Other passages in typical first-year university mathematics textbooks (e.g., college algebra, precalculus, and calculus) include both notation and words such as the following (from Larson, Hostetler, & Edwards, 2002, p 174):

> A function f is **increasing** on an interval if for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) < f(x_2)$.

Students often find the syntax of such sentences confusing. It is different from that of ordinary text; it contains abstractions; and there are few clues to the meaning of less familiar vocabulary or symbols (Shepherd, 2005). For example, the "if" in the above sentence should be interpreted as "if and only if," and the sentence is implicitly universally quantified. In addition, x_1 and x_2 are variables that do not have a special status because of the subscripts, as many students suppose.

1.2 Reading Comprehension Research

During the past forty years, conceptual shifts have led reading researchers to view reading as an active process of meaning-making in which readers use their knowledge of language and the world to construct and negotiate interpretations of texts in light of the particular situations within which they are read. (Borasi, Seigel, Fonzi, & Smith, 1998; Brown, Pressley, Van Meter, & Shuder, 1996; Dewitz & Dewitz, 2003; Flood & Lapp, 1990; Kintsch, 1998; McNamara, 2004; Palincsar & Brown; 1984; Pressley & Afflerbach, 1995; Rosenblatt, 1994; Schuder, 1993; Siegel, Borasi, Fonzi, Sanridge, & Smith, 1996). These conceptual shifts have expanded the notion of reading from that of simply moving one's eyes across a page of written symbols and translating these symbols into verbalized words, into the idea of reading as a mode of thinking and learning (Draper, 2002).

Current discussions of reading focus on how the reader creates meaning as a result of the interaction, or transaction, between the text and the reader (Flood & Lapp, 1990; Pressley

& Afflerbach, 1995; Rosenblatt, 1994). Reading researchers have found that competent readers actively construct meaning through a process in which they interact with the words on the page, integrating new information with preexisting knowledge structures (Flood & Lapp, 1990).

Reading and literacy researchers agree that reading includes both decoding and comprehension. Research on comprehension indicates that there are several strategies that good readers employ before, during, and after they read. These strategies seem to vary from reader to reader and depend on the material being read and the goals of the reader (Borasi et al., 1998; Brown et al., 1996; Flood & Lapp, 1990; Fuentes, 1998; Palincsar & Brown; 1984; Pressley & Afflerbach, 1995; Siegel et al., 1996).

One of the most comprehensive metastudies of reading research was conducted by Pressley and Afflerbach (1995). They developed a framework called Constructively Responsive Reading (CRR) that effectively combined the frameworks of many previous reading researchers (Rosenblatt, Brown, Kintsch, and others). They noted about 330 different activities that readers reported, or were observed, doing while reading. They produced a "Thumbnail Sketch" of the CRR framework that categorized activities of good readers into fifteen constructive responses. We have further reduced these responses to the following eight that we call *strategies*.

Table 1 Eight basic strategies for reading comprehension obtained from the CRR framework

1.	Preview the text to be read before reading to gain an overview and to make predictions about the text.
2.	Pay greater attention to information perceived as most important.
3.	Activate prior knowledge, integrate reading within text and with prior knowledge to interpret the text, construct meaning, and revise/adjust prior knowledge.
4.	Make inferences about information not explicitly stated.
5.	Determine the meanings of new or unfamiliar words.
6.	Monitor comprehension and change reading strategies if needed.
7.	Evaluate the text, remember text, and reflect on it after reading
8.	Anticipate the use of the knowledge gained.

In addition to the above strategies derived from the CRR framework, we found four slightly different collections of reading strategies. Each includes some subset of the eight strategies noted above. The four strategies are: Reciprocal Teaching (RT) (Palincsar & Brown, 1984); Transactional Strategies Instruction (TSI) (Brown et al., 1996); Self-Explanation Reading Training (SERT) (McNamara, 2004); and Transactional Reading Strategies (TRS) (Borasi et al., 1998). RT, TSI and SERT have been used for general reading, not the technical reading of a mathematics textbook. RT has been used with middle-school age poor readers with great success; TSI has been used successfully with second graders; and SERT was used with limited success with undergraduates.

TRS has been used with high school students; it employed texts, mainly essays about mathematics, but not passages taken from textbooks. TRS appears to be the only collection of strategies using activities to help student readers integrate and construct knowledge. In one such strategy, students used a string, thumbtack, and pencil to construct and explore circles—an activity that is said to have increased their understanding greatly. It would be interesting to investigate if, or how, such strategies could also support the reading of more technical mathematics textbooks, but such an investigation is beyond the scope of this paper.

Other research (Fuentes, 1998) has indicated that some difficulties in comprehension can be traced to an inability to integrate what is read with prior knowledge. The causes of this could range from insufficient prior knowledge and an inability to add to it, to adequate prior knowledge but an inability to integrate what is read with that prior knowledge. However, insufficient prior knowledge, in and of itself, does *not necessarily* lead to difficulties in comprehension. For instance, a reader with good algebra skills and knowledge, who has no background in, or prior knowledge of, complex numbers, might have no difficulty reading a beginning passage about complex numbers – provided there were somewhere he/she could look for additional background.

1.3 Potential Difficulties in Reading Mathematics

Combining what has been learned in reading research (Section 1.2) with some of the differences noted about mathematical text (Section 1.1), one might suspect that difficulties in learning from reading mathematics textbooks might include: (1) dealing with insufficient prior knowledge that comes from underdeveloped concept images; (2) dealing with the syntax and precision of mathematical definitions, examples, and exposition in mathematical writing; and (3) grounding the abstractness of mathematical ideas in concrete objects or actions while reading.

When reading mathematical text, there appears to be little room for an acceptable interpretation of a passage that is different from the one intended by the author. However, Edwards and Ward (2004) indicated that formal definitions are not used by students as much as their concept images when reasoning about the abstract ideas encountered in a typical upper-level mathematics class such as abstract algebra. This dependence on concept images may also occur for lower-division courses, and the concept images of different readers may contain different examples or procedures. This suggests that reading strategies for mathematics should advocate that students actively engage in working from their concept images to the actual definitions, and vice versa, in order that they come to a reasonable semblance of the meaning intended by textbook authors (Pinto & Tall, 2002).

We conjecture that some reading strategies that can help comprehension when reading, say, a history textbook might also work when reading a mathematics textbook. However, emphasizing strategies that "engage" students in activities that can help create "hooks" to prior knowledge, ground concepts in concrete objects or actions, or encourage readers to use the stated definitions as opposed to their individual concept images may be more important when reading a mathematics textbook. Since mathematics textbooks often do not have many clues, beyond the definitions themselves, to the meanings of less familiar terms and symbols, students need to be especially active in attempting to understand definitions and in monitoring their comprehension.

2. Research Questions

Although mathematics textbooks differ considerably from other texts (Section 1.1), what matters in the context of this study is the degree to which such textual differences actually influence the effectiveness of students' reading. Thus we ask: (1) Can first-year university students read their mathematics textbooks effectively? That is, (2) do they benefit mathematically from their reading? Here we took the point of view of an old proverb, "the proof of the pudding is in the eating." We examined whether our students could successfully carry out straightforward tasks (sometimes called examples, exercises, or problems) immediately after reading passages explaining how the tasks should be carried out, and with the passages still available.

We also considered our students general mathematical preparedness, as indicated by their ACT mathematics scores,[7] and their general reading ability as indicated by both their ACT reading scores and their use of CRR strategies. Because our students were good general readers and good at mathematics, any inability to properly complete a task should reflect mainly difficulties in reading specifically mathematics textbooks.

(3) In addition we ask what students do when reading and what obstacles they encounter? This includes examining what reading strategies our students are using. And are these strategies adequate? We begin to answer these questions by observing our students while they read aloud passages from their textbooks and worked straightforward tasks based on those readings.

3. Methodology

3.1 The students

The eleven precalculus and calculus students in this study attended a U.S. mid-western comprehensive state university at which they took all their coursework. The university has a student body of 6,500 students of which 5,500 are undergraduates. It has a moderately selective admissions standard. Six of the students were university age students. Five were students in a mathematics/science magnet secondary school located

[7] The ACT (American College Test) is a university admissions examination that includes a mathematics portion and a general reading portion.

on the campus of the university. Eight of the students were female, none were minorities. The mathematics courses taken by the eleven students carried normal university credit and were taught by a member of the regular university faculty -- the first author.

Students for the study were selected from a precalculus class of 17 (the secondary magnet school students) and from two sections of Calculus I with 41 students total. In the fourth week, we identified 33 good readers (12 precalculus, 21 calculus) with an ACT reading score ranging from 24 (70^{th} percentile) to 36 (99^{th} percentile). Based on the instructor's judgment, nine students (4 precalculus, 5 calculus) were eliminated because it appeared they had no problems reading mathematics and may have seen the material in previous courses. Of the remaining twenty-four students, eleven (5 precalculus, 6 calculus) volunteered to participate in this study. Ten of the students received a small amount of extra credit for participating in the study. The amount of extra credit received did not change any final grades. One calculus student dropped the class before the fourth week, but agreed to participate. That student was grouped with the precalculus students since that was the passage she read for the study.

The average reading ACT score for the eleven students was 28.6 (the median, 28, corresponds to the 87^{th} percentile) which compares favorably to the university average reading ACT score of 22.3 for incoming first-year students. For these eleven, the reading ACT scores were further broken into Social Studies/Science where their subscores ranged from 12 (68^{th} percentile) to 17 (98^{th} percentile) and Arts/Literature where their subscores ranged from 12 (63^{rd} percentile) to 18 (99^{th} percentile).

All but two volunteers — both calculus students who were not first-year university students—had ACT mathematics scores ranging from 23 (71^{st} percentile) to 30 (96^{th} percentile). The average mathematics ACT score for the eleven students was 25.2 (median score 27) which compares very favorably with the university average mathematics ACT score of 20.9 for all incoming first-year students. Thus, according to their ACT scores, these students were good students; that is, they were good at mathematics and were good readers of both the Social Studies/Science and Arts/Literature portions of the ACT.

3.2 The Courses

From the beginning, both the precalculus and calculus courses from which the students were chosen had a strong emphasis on reading their mathematics textbooks. The students were given handouts about reading mathematics on the first day of class, and beginning the second class period, students were given reading guides for use with the first several sections of their mathematics textbooks. An example of a reading guide and additional information about the teaching practices of this instructor appeared in Shepherd (2005).

During the first two weeks of the courses, all 58 students from the pre-calculus and calculus classes participated in a diagnostic interview as part of the instructor's normal teaching practice. This consisted of reading one of four short (one-half to two page) passages on partial fractions, algebraic vectors, absolute value, or symmetry. Students at this level are unlikely to be familiar with readings on these topics, but will normally find them accessible. After reading the short passage, each student was asked to complete a task, based on the passage read. In addition to being used diagnostically in teaching, these interviews served to familiarize the 11 subsequent volunteers with the interview procedures that they would experience later.

3.3 The Conduct of the Study

During the sixth and seventh weeks of the courses, the volunteers each selected a 90-minute time slot during which they were asked to read aloud a new section selected by the instructor/researcher from their respective textbooks. Five calculus students read Section 3.1 entitled "Extrema on an Interval" in Larson et al. (2002, pp. 160-164). The one calculus student who had dropped the course and the five pre-calculus students read Section 5.1 entitled "The Wrapping Function" in Barnett, Ziegler, and Byleen (2000, pp. 336-343). Along with definitions, theorems, examples, figures, and discussions, the calculus book has "Exploration" and the precalculus book has "Explore/Discuss" tasks to encourage students to become active as they read.

The students were stopped at intervals during their reading and asked to try a task based on what they had just read, or asked to try to work a textbook example without first looking at the provided solution. The calculus students were stopped an average of eight times (a maximum of nine times, a minimum of seven times). The precalculus students were stopped an average of three times (a maximum of four times, a minimum of three times). The tasks were straightforward; that is, they were based directly on the reading and required very little in the way of genuine problem-solving skills. The reading pages along with the interruptions and requested tasks appear in Appendix A for precalculus and in Appendix B for calculus. For example, for the reading from Larson et al. (2002), the calculus students were asked to determine from a graph whether a function has a minimum on a specified open interval. From Barnett et al. (2000), the precalculus students were asked to find the coordinates of a circular point, that is, a point such as $W(\pi/2)$ on the unit circle given by the wrapping function. After the entire section had been read and a few final tasks were attempted, the students were questioned about how reading during the interview differed from their normal reading of their mathematics textbooks (Appendix C).

All interviews were audio-recorded and transcribed. The interviewer also made notes during the interviews. The written work produced by the students during the interview was collected. The first author listened to the recordings carefully at least three times, making additional notes. These additional notes, along with the notes taken during the interview, were compared with the transcripts to create Tables 2 and 3 below.

4. Observations, Difficulties, and Strategies

4.1 Use of the CRR Strategies

Table 2 indicates that, for the most part, the students were attempting to employ the CRR strategies characteristic of good readers. It confirms that they were, in general, good readers, as indicated by their ACT reading scores (Section 3.1). In Table 2, for each of the eight CRR strategies (Section 1.2), we provide examples of observed behaviors, together with the number of students exhibiting that behavior. For example, six students read the title of the section, the introduction, or the caption at the start of their reading.

Table 2. Observed CRR activities of the students in this study.

CRR strategy (shortened)	Number of students observed	Examples of Observed Activities, Behaviors, and Comments
1. Overview before reading.	6	Read titles, introduction or outline captions at the start of the reading.
2. Look for important information and pay greater attention to it.	11 3 2	Reading selectively, slowing down, pausing and rereading sentences. Specifically stated something like, "This must be important." In questioning at the end, reported only looking at the examples.
3. Attempt to relate important points in text to one another.	7 11	Tried to relate a point in the current reading to earlier points. Looked at tables or went back in the reading to reread previous parts.
4. Activate and use prior knowledge to interpret text.	6	Students did not activate prior knowledge before reading but were observed recalling things learned in previous mathematics courses while reading.
5. Relate text content to prior knowledge.	7	Specifically related what they read to something in their prior knowledge.
6. Reconsider or revise hypotheses about meaning of text.	5	Showed that they had revised their understanding of the text by the end of the reading.
7. Reconsider or revise prior knowledge based on text.	None	There were no overt observations of the changing of prior knowledge, however this does not mean students did, or did not, do this.
8. Attempt to infer information not explicitly stated.	11 1	Tried to fill in details and give reasons while reading the examples. Filled in a reason incorrectly and subsequently corrected his reasoning.
9. Attempt to determine the meaning of unfamiliar words	11	Recognized when something was not understood and many tried different strategies, such as rereading definitions or paraphrasing, hoping to determine some meaning.
10. Use strategies to remember text.	11 4 1 3 11	Repeated or reread parts of passage. Wrote notes or copied important ideas onto paper. Seemed to create a concrete visualization of a concept; for instance, comparing the wrapping function to a ribbon. Constructed analogies, identifying the u-v coordinate system as the x-y coordinate system. Paraphrased, though, not always correctly.
11. Change reading strategies when comprehension is not proceeding smoothly.	3	Stated they would "go ask for help."
12. Evaluate the qualities of text.	7	Several students had specific comments about the text in the debriefing (see Appendix B) related to the appropriateness of examples, the clarity of the author, etc.
13. Reflect on text after text has been read.	11 3	Gave some indication of reflecting on the text while reading. Specifically recognized some unresolved understanding at the end of the reading.
14. Carry on responsive conversation with the author	3	Several students commented on "what the book wants" while reading or working examples.
15. Anticipate the use of knowledge gained from the reading.	2	Anticipated how the reading would be used in an application.

4.2 Difficulties in working tasks

All of the students in our study had considerable difficulty correctly completing some of the straightforward tasks based on their reading. The percent of tasks done correctly by individual students ranged from 13% to 76%. The tasks were what might be called "routine exercises." For instance, after the textbook had defined the wrapping function, *W*, and had explained the calculation of the exact values for $W(0), W(\pi/2), W(\pi)$, and $W(3\pi/2)$, the routine exercise given was: *Find the coordinates of the circular point* $W(-\pi/2)$. A similar explanation was given for values of the wrapping function at integer multiples of $\pi/4, \pi/3$, and $\pi/6$, followed by routine exercises. Five of the six students who read the precalculus passage did not find correct values of the wrapping function in two or more instances. Also, four of the five students who read the calculus passage containing the definition of extrema of a function on an interval could not determine from its graph whether the function had a minimum. Ten of the students stated at some point that they did not understand something and yet made no attempt to locate the source of their confusion. Five students, three precalculus and two calculus, gave up at some point. They stated that they had no idea what to do either while trying to work on a task or when reading through an example with a solution. When questioned, one calculus student stated she would just move on, the other four stated they would quit and ask for help before continuing. However, they continued to read at the request of the interviewer. Table 3 gives information on the number of tasks attempted, done correctly, incorrectly, or not done, by each student and the number of different CRR strategies each student was observed using.

Table 3. Correctness of tasks and number of CRR strategies observed.

Student	# tasks attempted	# Correct (% correct)	# Incorrect	# not done (skipped or gave up)	Incom-plete	Read/ not worked	Read as worked	"correct" w/wrong reasons	# CRR strategies observed
Precalculus									
Alicia [8]	19	9 (47%)	5	5					7
Bryan	18	9 (50%)	4	2	1	2			8
Christie	21	3 (14%)	7	7	1	2		1	10
Darcy	8	1 (13%)	2	2	1	2			12
Ellis	17	13 (76%)	2	1	1				11
Faye	20	6 (30%)	6	7		1			8
Calculus									
Tara	22	8 (36%)	2	2	4	5		1	9
Vannie	22	12.5 (57%)	2.5	1		2	4		11
Winnie	22	10.5 (48%)	1.5	3	1		6		11
Yates	22	8 (36%)	4	2	2	1	5		10
Zoe	23	8.5 (38%)	6.5	1	1	1	5		8

In Table 3, there appears to be little or no relationship between the number of good reading strategies observed and the percent of correctly performed tasks. However, good reading strategies might have been present without having been observed. Also, good readers may favor different strategies in different situations (Pressley & Afflerbach,

[8] Students' names are pseudonyms.

1995). This still leaves open the question, why good readers, doing the things good readers are supposed do, had the number of difficulties exhibited in Table 3.

There is a pattern in Table 2 that suggests where to start answering this question. The CRR strategies related to activation, use, and revision of prior knowledge (Strategy 3) and the CRR strategies related to an evaluation of what has been read (Strategies 6 and 7), though observed in some of the students, were not generally favored. This lends some support to the idea that some of the difficulties related to reading mathematics might include dealing with insufficient prior knowledge and the evaluation of what has been read. Such evaluation may depend to some extent on understanding the precision in meaning intended in mathematical writing, and the grounding of abstractions in concrete objects or actions (Section 1.3).

Next we discuss some of the specific difficulties observed. We present observations related to: (1) understanding definitions; (2) use, or non-use, of theorems; (3) consideration and use of examples encountered during reading; (4) doing unguided explorations; and (5) reading expository passages.

4.2.1 Understanding definitions.

In mathematical writing, it is intended that everyone who reads the definition of a concept with comprehension will have essentially the same basic understanding of that definition. Different individuals' concept images need not agree, but everyone should be able to agree on whether or not an example satisfies the concept's definition.

For the calculus students, one difficulty appeared to come from an inadequate concept image of the word "function." After reading the definition (Appendix A), Vannie[9] was asked to look at the graphs of eight functions and determine whether they had minimum values. As she looked at the graph of the first one, *51a*, a piecewise continuous function, she went back to the definition and tried to compare the graphical information with the definition.

> "You're on the interval I as they designate. You're supposed to look at […] is it c or x they use. … for all the x's, $f(c)$ is supposed to be your minimum point. Well, $f(c)$ on this portion is your minimum point is a real number, but on this one it is not because it is open. So, if you look at it from [...] since it's totally two different things coming in. I don't know if you say well this one does have a minimum and this one doesn't or if they go together, then they don't. I don't [...] that part I [...] I'm not clear on." [10]

She came to no resolution. Vannie's difficulty seemed to arise from her inadequate concept image of function. Although she clearly tried to use the definition of extrema, she did not appear to recognize the graph of a function that has a jump discontinuity as a single function.

[9] All names are pseudonyms. Names starting with letters at the beginning of the alphabet are precalculus students and names starting with letters at the end of the alphabet are calculus students.
[10] When students are speaking, their comments are shown in regular typeface; when they are reading text they are shown in italics; pauses are shown as […] and … indicate omissions.

A second difficulty arises from not carefully re-reading a definition, but relying entirely on memory. Christie read about the wrapping function and how to calculate its values for integer multiples of $\pi/2$, orally answered two worked examples incorrectly (with the work hidden), and read their solutions. She then tried to answer the first matched problem, find the coordinates of $W(-\pi)$. She said, "It's going to be (1,0) because you're going . . . up π every time, every quarter of a circle…. So if we just start at the top [(0,1)] and then go down one π, I think we'd be at (1,0)." Not only did Christie start wrapping at the wrong point, but she did not understand that the measure of a quarter of a circle is only $\pi/2$. She never went back to the definition to check her starting point. Later in her reading, she discovered the starting point was (1,0) instead of (0,1). At the end of the interview, when asked if there was any notation that bothered her, she was still confused about the starting point. She said, "And I still don't [...] I mean they still start you at the *v*-axis sometimes, and they start you at the *u*-axis sometimes, I think. So, I'm not real sure on that aspect of it."

A third difficulty related to definitions arises from not distinguishing between definitions with similar wording, such as relative extrema versus absolute extrema. One of the tasks given the calculus readers included the directions, *Determine whether the function has a relative maximum, relative minimum, absolute maximum, absolute minimum, or none of these on the interval shown.* (Larson, et al., 2002, p. 165) Zoe worked through the exercise, looked up the definition of extrema on an interval which included absolute extrema, but not relative extrema (Appendix B). In the debriefing (Appendix C), she was asked if there were any words that bothered her; from her comments one can see that she had not been careful to distinguish between definitions of related concepts. Below she refers to the exercise whose directions are given above.

> "It said to find any relative minimum, relative maximum, absolute minimum, and absolute maximum. But in the first of it [definition of extrema], they said that those are the same things. So I wasn't quite sure why they were asking me to find possibly four different things if they're supposed to be just the same thing, but synonyms. If there's something different they need to be more clear about that and […] I thought maybe one of them was dealing on an open interval and one of them was dealing on a closed interval, but since I didn't know, I just went under the assumption that they're the same thing."

4.2.2 The use, or non-use, of theorems.

The students in this study also had some difficulties related to theorems encountered in the textbook. Some students could not assign the correct authority to a theorem and some had difficulties understanding the implications of a theorem. The calculus students read The Extreme Value Theorem, followed by an Exploration (Appendix B). They were then asked to answer a true/false question: *If a function is continuous on a closed interval, then it must have a minimum on the interval.* One student, Tara, answered the true/false question correctly and correctly gave the Extreme Value Theorem as the reason. Winnie answered true, but gave no reason, so we cannot tell whether her reasoning was correct or not. The other three calculus students had errors in their answers. Two examples are given below.

One student, Vannie, tried to use the definition of Extrema on an Interval, but could not use it to answer the true/false question. She consulted the definition of Extrema on an Interval, and then "pled the fifth" because she did not know the answer. Vannie seemed to understand that the definition was not enough, but did not know where else to look, even though she had just read the Extreme Value Theorem.

Two errors combined when a student, Zoe, relied on a visual part of her visual concept image that incorrectly indicated that a horizontal line had no extrema, and also confused an implication with its converse. Zoe justified her incorrect answer with an example that seemed to show visual reasoning as opposed to using the Extreme Value Theorem. She said, "That's false, [...] because it doesn't, it's not continuous. [...] *If a function's continuous on a closed interval*, [...] well, it need not, but I can't think of an example. If it were a line it wouldn't necessarily have a minimum. I guess that will be my example." She was unaware that her answer was wrong.

4.2.3 Consideration and use of examples.

Considering examples in a textbook can be thought of as a way to enrich one's concept image. Generally, examples are of three types and serve several purposes: (1) They allow a reader to check whether a concept or calculation has been understood. For instance, after developing the wrapping function for multiples of $\pi/2$, the textbook asks readers to find the coordinates of $W(-\pi/2)$ and $W(5\pi/2)$. Similar requests follow the discussion of the wrapping function for multiples of $\pi/4$, $\pi/3$, and $\pi/6$. (2) Examples can be used to prepare for upcoming concepts and definitions. For instance, after the definition of Relative Extrema and before the definition of Critical Number, the textbook asks readers to find the value of the derivative at each relative extrema of three functions, given their graphs and apparent relative extreme points. (3) Examples demonstrate/illustrate a concept and provide practice. For instance, the calculus textbook gives the steps, and the reasoning behind them, for finding extrema on a closed interval and then asks readers to practice these on polynomial, rational exponent, and trigonometric functions.

In the interviews, students were asked to solve each worked example with the textbook solution covered with a Post-it© note . Most of the precalculus readers could work through most of the examples without looking at the solution. These examples were all of type (1) above and involved only one or two reasoning steps. After working the example themselves, some students would read the textbook's solution thoroughly, while others would only skim it. One precalculus student, Christy, did not write down her answers so upon checking the solution, was unaware of errors she had made. Two precalculus students, Faye and Darcy, had difficulties with fractions, such as recognizing that $3\pi = 6\pi/2$.

None of the calculus readers was able to complete the final three examples in their textbook passage, without looking at the solutions provided or comparing their work with that of the book. These examples were all of the third type above and concerned finding the extrema of a trigonometric function and estimating extrema for graphically presented functions. These examples required more than one reasoning step and more algebra to complete than did the precalculus examples.

The textbook provides a list of steps to follow in order to find the extrema of a function on a closed interval. Although the calculus students tried to follow the steps, three of them had difficulty with algebraic concepts (negative exponents, factoring, trigonometric identities), and all of them gave up trying to figure out the trigonometric example. Even though the textbook clearly lays out the steps needed to find the absolute extrema on an interval, no student completed the three worked examples in the passage without great assistance from the textbook solution. Although they continued to read for the interview, two students stated they would normally give up before reaching the final example. Vannie indicated she would ask her group for help before continuing, and Tara indicated she would ask the teacher about the example in the next class period.

Another difficulty occurred when students did not pay close attention to relevant definitions as they worked examples. As noted in Section 4.2.1, Christie did not pay close attention to, nor did she look up, at what point the wrapping function starts, and because of this, she worked the first two examples incorrectly.

Another difficulty occurred when one precalculus student, Faye, focused on the development of the wrapping function and tried to derive its values for multiples of $\pi/4$ directly, rather than using symmetry as suggested by the textbook. Both of her attempts indicated that while she might have understood the basic algebra, she did not see how to apply it to different quadrants. Of the five precalculus students who read this passage, she is the only one who did not use symmetry. Although this might be an example of "folding back," noted by Pirie and Kieren (1994), Faye seemed very interested in showing she could derive the values directly.

Faye first read the algebraic development of the coordinates of $W(\pi/4)$. Then, just before an example to work, she read: *Using the symmetry properties of a circle, the unit circle is symmetric with respect to both axes* [She repeated this phrase.] *and the origin, we can easily find the coordinates of any circular point that is reflected across the vertical axis, horizontal axis, or origin from* $W(\pi/4)$.

Faye then read the directions to Example 2. "*Find the coordinates of the circular points* A. $W(5\pi/4)$ *and* B. $W(-\pi/4)$. … Let's see, one, two, three, four, five […] I don't think so […] there's nothing for me to count. [...] there's no axis there […] so […] I don't know if I [...] if my counting would be equal … I didn't know, if I would still like would count like in one ... like in, like that, I don't know, just because there's nothing to count on I don't think OK, alright." She started to rederive by writing $a^2 + b^2 = 1$, then read the solution to part A, mostly silently. From her body language, she seemed to understand it, but did some deriving. The answer she wrote was $(-1/\sqrt{2}, 1/\sqrt{2})$ which was incorrect. She read the solution to part B concerning $W(-\pi/4)$. " … *(1,0) we proceed one-eighth the way around the unit circle in a clockwise direction…the fourth* *quadrant*…wait.. ok..that's right…*on the circle halfway between (0,-1) and (1,0) as indicated in Figure 6* [in the textbook]. This was followed by silence and low whispering. She rederived the values during this silence. "That works. Ok." She had written the answer to Example 2B as ($-1/\sqrt{2}, -1/\sqrt{2}$), which was incorrect. When Faye tried the matched exercise 2A, which was her third attempt to calculate one of these values, she apparently did use symmetry to find her answer. At the end of the interview, when Faye tried to find the value of the wrapping function at $\pi/6$, she correctly rederived the wrapping function at $\pi/4$, instead of finding W($\pi/6$).

The students, particularly those in the calculus group, seemed to find it difficult to work the examples and reconcile their work with that shown in the textbook. At the end of the interview, Winnie said, "A lot of the times their examples are the easier problems and then the ones you see in the lesson are […] (shrugging)." It may have been that the passage selected for the interview had especially difficult examples, but it seems more likely that the examples in each section of the textbook progress from easy to more difficult, and that students sometimes only attempt to understand the easiest ones.

Perhaps not surprisingly, the two students with the lowest ACT mathematics scores 16 and 20 (23rd and 55th percentile, respectively), Tara and Vannie, had great difficulty completing the required algebra and in explaining the solutions given in the calculus textbook passage. Their incomplete prior knowledge of algebra caused them difficulties. Vannie's incomplete prior knowledge of negative fractional exponents caused her to become frustrated and give up attempting to understand the calculation. She tried to work Example 4 that asked the reader to find the extrema of $f(x) = 2x - 3x^{2/3}$ on the interval [-1,3]. Vannie attempted to take the derivative and set it equal to zero. She wrote $f'(x) = 2x - 2x^{-1/3} = 0$. At this point she checked the solution to confirm her derivative and said, "They did something crazy. Ok. What did they do? […] I'm confused. . . . I don't understand their math or their […] what they did. … I figured it was just a basic [...] you did the derivatives in the subtraction." She eventually fixed her derivative but still could not get to the simplified derivative shown in the textbook, which was $f'(x) = 2 - \frac{2}{x^{1/3}} = 2\left(\frac{x^{1/3} - 1}{x^{1/3}}\right)$. The negative exponent confused her, even though she had tutored college algebra in the past. Her final comment after reading through the entire solution was, "At this point, if I was really reading this I would be frustrated and quit and then I would go ask somebody."

The difficulties that occurred when students tried to work the examples in the textbook were from (1) not applying the definition correctly, (e.g., starting at the wrong point on the unit circle when wrapping), (2) trying to rederive the coordinates because of paying more attention to the detailed algebraic development and less to concepts like symmetry, or (3) having some weakness in prior knowledge, either procedural (algebraic) or conceptual.

That some students could not carry out these tasks does not necessarily indicate that they got nothing from their reading. Some of the students "learned" from their mistakes. Bryan and Alicia, both precalculus students, had some errors in working the examples and the matched problems. There were twelve tasks, each to find the wrapping function value for some multiples of $\pi/2, \pi/3,$ or $\pi/4$. Each student did six correctly and four incorrectly. Brian then read through the other two without appearing to work them, and Alicia skipped two tasks at the bottom of the page. However, at the end of their reading, each was asked to complete some tasks similar to the ones they had done while reading, and each completed these tasks correctly.

4.2.4 Doing unguided explorations.

A feature of both textbooks used by the students in this study is a more open-ended type of task, called an "Exploration" or an "Explore/Discuss" task, where no guidance is given and an explanation may be required. The students were given the option of doing the Exploration (with two parts) in the calculus reading and the two Explore/Discuss tasks in the precalculus reading (Appendices A and B). Four of the five calculus students chose to do the first part of the Exploration, and one, Tara, also did the second part. Tara came to a wrong conclusion on the first part, saying that there was no maximum for a quadratic function on a closed interval. She said, "I think it's infinity because the graph keeps going and I can't see any point." On the second part with a cubic on the same closed interval, she said, "I think the minimum and maximum of both of these is infinity since I can't find an ending point on either one of them." Having read The Extreme Value Theorem just prior to this did not lead her to see a conflict between the theorem and her answers. However, as noted above (Section 4.2.2), Tara answered correctly, with a correct reason, the true/false question posed immediately after this, *If a function is continuous on a closed interval, then it must have a minimum on the interval.* Zoe chose not to do the Exploration because "…that's not going to help me." Most of the precalculus students chose not to do the Explore/Discuss features either with comments such as, "I don't understand what they want me to do," from Christy, or "They might think that's an effective memory aid but that's confusing me so I'm moving on" from Ellis.

4.2.5 Reading expository passages

One of the purposes of exposition in mathematics textbooks is to help students integrate definitions, theorems, and examples with prior knowledge. We did not observe specific difficulties related to the exposition. All students read the expository parts of the textbook since it was part of the interview, but upon questioning at the end of the interview some students viewed exposition as less important -- something they often skipped or skimmed. Students wanted to concentrate on problems and find examples similar to the problems given in the text, often ignoring the exposition that tied together concepts and examples. Some of the student comments included: "I learn by examples."—Winnie. "Sometimes it's just jibberish, but stuff that they mean to attempt to stand out then I read that, but usually, at the beginning of the chapter I try not to read. I just read the definition because otherwise it's just confusing."—Zoe. "It takes quite a while to read through [the chapter] like that, too, maybe an hour, hour and a half."—Yates.

5. Discussion

Our students could not read their mathematics textbooks very effectively, as indicated by their inability to properly carry out many of the straightforward tasks in the reading interviews. Correctly working most such tasks is an indication of the ability to use knowledge and understanding essential in carrying out more complex later tasks that are a major component of the students' courses. Only three of our eleven students (Bryan, Ellis, and Vannie) could work at least half of the tasks, and only one of them could work three-fourths of the tasks (Table 3). Because our students were good mathematics students according to their ACT mathematics scores (Section 3.1) and good general readers according to both their ACT reading scores (Section 3.1) and their use of the CRR strategies (Tables 1 and 2), we suggest their inability to carry out the tasks arose largely from an inability to read mathematics textbooks in particular (as opposed to a general inability in mathematics or reading). This agrees with the students' own views, that is, they believe they do not benefit from reading major parts of their mathematics textbooks, and often avoid doing so (Section 4.3).

5.1 *Why is Effective Reading of Mathematics Textbooks Difficult?*

We suspect that several factors may have contributed to our students' ineffective reading. In Section 1.1, we pointed out a number of ways that reading mathematical text can differ from reading other text, and such differences might in some situations contribute to ineffective reading. However, most of the differences pointed out in Section 1.1 do not occur in the passages our students read, and what differences were there did not often cause our students to stumble in reading. For example, they could read equations and the notations for functions, intervals, and points. Thus we have looked further for factors contributing to our students' ineffective reading.

5.2.1 The necessity of cautious reading

There is a striking difference between our students' reading and the way many mathematicians read similar passages. By reading similar passages, we mean reading whose purpose is to construct new knowledge that is immediately testable procedurally – knowing how to reliably carry out small associated tasks such as measuring an angle when considering the wrapping function or finding the maximum of a function.

From our experience as mathematicians, we suggest that most mathematicians read the above kind of material in what might be called a very cautious way. They are aware of the precise nature of mathematical writing. Thus, they tend to look for hints of their own misunderstandings by carrying out, and evaluating their performance on tasks provided by the author, and even by occasionally inventing and working additional tasks. When an error is detected, mathematicians are likely to reread the associated passage and rework the task until their error is understood and corrected. We further suspect that this kind of careful, usually slow, reading is based on individuals' (perhaps tacit) knowledge that, in mathematics, neglected small errors are likely to lead to significant later errors, and that one's own reading occasionally generates such small errors.

In contrast to the above picture, our students' reading was quite incautious. Ten of the students stated they did not understand something during the interviews, but made no attempt to determine what was causing their confusion, and five gave up at some point (Section 4.2).

5.2.2 *Main reasons for our students' difficulties*

There seemed to be at least two main reasons for the observed student difficulties: (1) Even good secondary and beginning university student, such as ours, often do not know how to use mathematical definitions and theorems. (2) Our students had difficulty integrating and connecting what they had read with their own prior knowledge, and found it difficult to adjust, update, and revise their personal concepts images to include new examples that were more in line with the newly introduced concept definitions. These difficulties presented obstacles for some students in this study, often resulting in frustration. They then expressed a desire to quit reading, to ask for help, or to skip those parts of the reading passages that they felt were not helpful. However, when someone was present to keep them on task, our students used many of the CRR strategies, although not always effectively.

Still most of our students seemed to benefit at least somewhat from their reading. This was observed for at least two students (Alicia and Brian), and was apparent to the teacher/interviewer for nearly all students when the content of the interview passages was subsequently discussed in class. However, at the time of the interviews, our students did not seem to perceive that they were benefiting from their reading, and many indicated that they would have given up had it not been an interview situation.

In the following section, we offer some suggestions that might be given to students to help them gain more from their reading.

5.2 Some strategies from the CRR framework to stress for reading mathematics

Our students' difficulties suggest that, for reading a mathematics textbook, parts of the CRR framework might need to be emphasized more than currently seems to be the case. For example, Strategies 1 and 2 (Table 1) recommend that before reading a passage, a reader should decide which parts to concentrate on, or perhaps even to attend to. For mathematics textbooks, such as those our students read, an unfortunate overly simplistic approach to Strategies 1 and 2 may be induced by portions of the textbook being visually set apart (Section 1.1).[11] Some of our students reported that they normally read mainly the examples and found the expository parts of their textbooks confusing, or considered them of minor importance. In this study, students read the expository parts only because they were part of the interview (Section 4.2.5). At first glance, such selective reading may seem consistent with Strategies 1 and 2, but it is so extreme as to be unhelpful. In reading a mathematics textbook, little material, except possibly for occasional historical notes, can be safely omitted. The expository part of a textbook can help a reader build a rich concept image for a concept whose definition might otherwise seem too abstract.

[11] Mathematics textbook authors probably set apart information such as definitions to encourage readers to look back to them when reading later passages. Such looking back to check details is a common practice among skilled readers of mathematics, such as the authors.

Rich concept images can be useful in bringing to mind needed ideas and examples when carrying out subsequent tasks.

In addition, Strategy 5 of the CRR framework (Table 1) indicates a need to determine the meanings of new and unfamiliar words. In reading mathematical text readers need to be aware of the difference between stipulated and extracted definitions, whereas in reading most nonmathematical textbooks, it is often possible to infer meanings from the context. Also, when reading most nonmathematical text, different readers can infer different, even conflicting, information and meanings because the definitions on which the inferences are based can often be interpreted in more than one way. However, if one uses considerable care and caution with the stipulated definitions of mathematics, it is possible for readers to be virtually certain that the inferences they draw are logically equivalent to the author's. Furthermore, for previously introduced technical terms, authors of mathematics textbooks assume their readers' meanings will be the same as their own. Thus, for reading mathematics, CRR Strategy 5 should include the idea that readers should be cautious in drawing inferences, but that with effort, readers can reliably arrive at the author's meaning.

Several of the difficulties observed in this study suggest our students were not following Strategies 4 and 5 (Table 1) with the degree of caution and preciseness required in mathematics. For example, Vannie (Section 4.2.1) became confused and could not proceed because the textbook referred to a split-domain function that she saw as two separate functions. It did not occur to her that her meaning of function might be incorrect. However, careful checking with the definition might have revealed to Vannie that her idea of function included a requirement (continuity) that the definition does not mention. Adding requirements is inappropriate when applying stipulated definitions. Thus, for mathematical reading, CRR Strategy 5 might need to include the strategy of carefully checking meanings with definitions. Such careful checking could also have benefited Christie (Section 4.2.1) who "remembered" the wrapping function started at the wrong point (0,1) and seemed to have decided incorrectly that the measure of a quarter circle is π. Including in CRR Strategy 4 the idea that it is not only permitted, but also very useful, to look back at definitions or theorems might prevent some incorrect inferences from being made.

Successful readers of mathematics must not only determine the meanings of words (Table 1, Strategy 5) or concepts, but must also be able to construct meanings and enhance their concept images by working through examples and nonexamples given in the textbook or by constructing and considering their own examples. The most successful reader, as indicted by the highest percentage of tasks completed correctly, was Ellis. He was the only student who, when he did not find the correct coordinates for $W(-7\pi/6)$, created his own example to make sure that he had understood the calculation. This is reminiscent of the results of Dahlberg and Hausman (1997), who found that among senior mathematics majors, those most able to understand and use a newly introduced abstract definition tended to, and were able to, generate their own examples.

Another mark of successful readers of mathematics is their approach to worked examples (tasks) found in the textbook. Often such worked examples involve computations, but they can also require visually inspecting a diagram and comparing parts of it with a definition. While such worked examples can be read or inspected passively, it is better for a reader to approach them as if they were his or her own work, or as if the textbook might contain errors. Many of the difficulties described in this paper surfaced when the interviewer asked students to take an active stance, working an example (task) before considering the textbook's solution. Students who develop a habit of reading mathematics textbooks actively will not necessarily avoid all difficulties; however, doing so can expose many difficulties.

The above two points, students developing a habit of constructing examples for themselves and of reading worked examples (tasks) actively, do not appear explicitly in the CRR framework and might be useful additions for mathematics readers.

6. Implications for Teaching and Future Research Questions

This study suggests that many first-year university students could benefit from some instruction in how to read a mathematics textbook. University mathematics instructors may need to encourage their students to become more active in reading. This might include getting students to do a better job of activating their prior knowledge, teaching students strategies to help them integrate what they are reading and learning with prior knowledge, getting students to approach definitions as stipulative rather than descriptive, and teaching students how to construct their own examples and nonexamples by carefully consulting the formal mathematical definitions of concepts.

Readers need to know how to "look up" definitions, and that it is "okay" to go back to definitions when reading mathematical text. This should carry with it the idea that a reader may need to adjust his or her concept image to be more in line with the concept definition. Also, readers need to learn to pay attention to each word in a definition since changing even one word can signal a difference between two concepts. While the students in this study did sometimes look up definitions, they frequently did not appear to, or could not, use that information correctly when attempting to carry out a mathematical task.

One approach might be to try to help students both with their tendency to and ability to work through and understand the mathematical tasks that typically follow immediately after the introduction of new ideas or techniques. Perhaps it would be helpful to assign, in each class, a brief passage of new material to be read in advance as homework. A small portion of the next class could then be devoted to helping students having difficulties with that passage. In that way, some of the difficulties we described (Section 4.2) might be identified and dealt with. Students might come to understand that it is appropriate to look back to previous definitions and theorems, and to be very careful about the meanings of words. For a more nuanced, but still practical, approach to helping students with reading their mathematics textbooks, see Shepherd (2005).

Turning to future research, how might the CRR framework be further developed and used for teaching? What additional specialized strategies are critical for understanding mathematics textbooks? In what ways would using such an extended framework in teaching reduce the kinds of student reading difficulties we have described?

It would also be very helpful to investigate student attitudes and beliefs about reading mathematics textbooks. Do some students believe that they cannot usefully read a textbook without help? Do many students believe that they will benefit most by reading mainly, or only, the worked examples? Do they feel it is worth attempting a task that is already worked out in the textbook?

Finally, this research looked at good readers, as indicated by their high ACT reading scores and their use of many of the strategies in the CRR framework, and found they are not necessarily successful at reading mathematics. What are the actual reading practices of more typical students? For example, do they go back to the details of a definition while attempting a task? Students' actual practices may differ from what they report doing.

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Appendices A and B include copies of the textbook pages that the students were asked to read. Permission to include these pages has been received from the publisher. The pages have been cut apart so that comments and tasks could be inserted to indicate when the students were asked to perform a task.

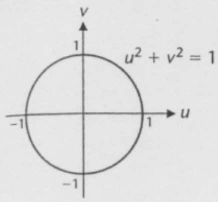
Appendix A contains the passage read by the precalculus students. All tasks that the precalculus students were asked to perform were contained within the selected pages.

Appendix B contains the passage read by the calculus students. In addition to the selected pages and indicated tasks, it also includes copies of the exercises that the students were asked to attempt.

Appendix A (Part I) —Precalculus reading passages with interruptions:

Definition of the Wrapping Function

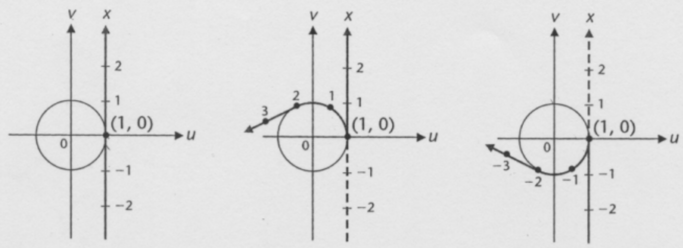
FIGURE 1
Unit circle.



The important *circular function* definitions presented in the next section are based on a function *W*, called the *wrapping function*, whose domain is the set of real numbers and whose range is the set of points on the *unit circle*. By the **unit circle** we mean the circle of radius 1 with center at the origin of a rectangular coordinate system. Its equation is $u^2 + v^2 = 1$ (see Fig. 1).*

In defining the **wrapping function**, we "wrap" a real number line with origin at (1, 0) around the unit circle—the positive real axis is wrapped counterclockwise, and the negative real axis is wrapped clockwise. In this way, each real number on the real line is paired with a unique point, called a **circular point**, on the unit circle, as shown in Figure 2.

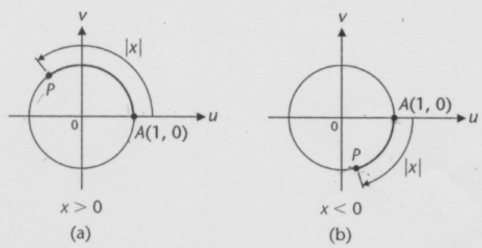
FIGURE 2
The wrapping function.



To locate the circular point associated with a number such as 37 or −105, the number line is wrapped many times around the circle.

An equivalent way of pairing real numbers with points on the unit circle is to think in terms of *arc length*, assuming we know what arc length is. To find the circular point *P* associated with the real number *x*, we start at *A*(1, 0) and move $|x|$ units along the unit circle, counterclockwise if *x* is positive and clockwise if *x* is negative. The length of arc *AP* is $|x|$ (see Fig. 3).

FIGURE 3
The wrapping function and arc length.



It is important to be able to find the coordinates (*a*, *b*) of the circular point *P* associated with a given real number *x* so that we can write $W(x) = (a, b)$. In general, this is difficult and requires the use of a calculator. However, for certain real numbers, integer multiples of $\pi/6$, $\pi/4$, $\pi/3$, and $\pi/2$, we can find the exact coordinates of the corresponding circular points using simple geometric properties of a circle.

*We use the variables *u* and *v* instead of *x* and *y* so that *x* can be used without ambiguity as an independent variable in defining the wrapping function in this section and the circular functions in the next section.

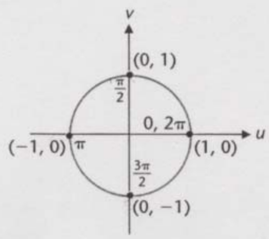
Exact Values for Particular Real Numbers

We start our investigation by finding the circumference of the unit circle. Since radius $r = 1$, the circumference is

$$2\pi r = 2\pi(1) = 2\pi \quad \text{Circumference of the unit circle}$$

One-fourth, one-half, and three-fourths of the circumference are, respectively, $\pi/2$, π, and $3\pi/2$. The circular points corresponding to these real numbers are on the coordinate axes, and hence, their coordinates are easily determined (see Fig. 4).

FIGURE 4
Circular points on the coordinate axes.



$$W(0) = (1, 0)$$
$$W\left(\frac{\pi}{2}\right) = (0, 1)$$
$$W(\pi) = (-1, 0)$$
$$W\left(\frac{3\pi}{2}\right) = (0, -1)$$
$$W(2\pi) = (1, 0)$$

Following the same procedure, we can find the coordinates of *any* circular point on a coordinate axis—that is, for any circular point corresponding to a real number that is an integer multiple of $\pi/2$.

EXAMPLE 1 Finding the Coordinates of Circular Points

Find the coordinates of the circular points.

(A) $W(-\pi/2)$ (B) $W(5\pi/2)$

The student readers were interrupted at this point and asked to work this example without looking at the solution, which was covered with Post-it© notes. Then each student was asked to read through the solution and work the Matched Problem.

Solutions

(A) Starting at (1, 0), we go one-fourth the way around the unit circle in a clockwise direction (see Fig. 4). Thus,

$$W\left(\frac{-\pi}{2}\right) = (0, -1)$$

(B) Starting at (1, 0) and proceeding counterclockwise, we count quarter-circle steps, $\pi/2$, $2\pi/2$, $3\pi/2$, $4\pi/2$, and ending at $5\pi/2$. Thus, the circular point is on the positive vertical axis, and we have

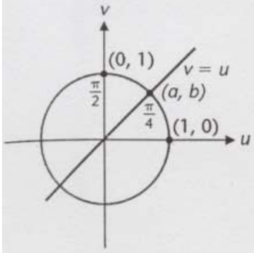
$$W\left(\frac{5\pi}{2}\right) = (0, 1)$$

MATCHED PROBLEM 1 Find the coordinates of the circular points.

(A) $W(-\pi)$ (B) $W(3\pi)$

The student was then asked to continue reading:

FIGURE 5
Circular point $W(\pi/4)$.



We now find the coordinates of the circular point $W(\pi/4)$. Since $\pi/4$ is one-half the arc joining (1, 0) and (0, 1), the circular point $W(\pi/4)$ must lie on the line $v = u$, as shown in Figure 5. Since $W(\pi/4)$ is on the line $v = u$ and on the circle $u^2 + v^2 = 1$, its coordinates (a, b) must satisfy both equations. That is,

$$a = b \qquad \text{and} \qquad a^2 + b^2 = 1$$

Substituting a for b in the second equation, we have

$$a^2 + a^2 = 1$$
$$2a^2 = 1$$
$$a^2 = \frac{1}{2}$$
$$a = \pm\frac{1}{\sqrt{2}}$$
$$a = \frac{1}{\sqrt{2}}$$

$a = -1/\sqrt{2}$ must be discarded, since $W(\pi/4)$ is in the first quadrant.

Using the first equation, we see that

$$b = a = \frac{1}{\sqrt{2}}$$

Therefore,

$$W\left(\frac{\pi}{4}\right) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

Using symmetry properties of a circle—the unit circle is symmetric with respect to both axes and the origin—we can easily find the coordinates of any circular point that is reflected across the vertical axis, horizontal axis, or origin from $W(\pi/4)$.

EXAMPLE 2

Finding the Coordinates of Circular Points

Find the coordinates of the circular points.

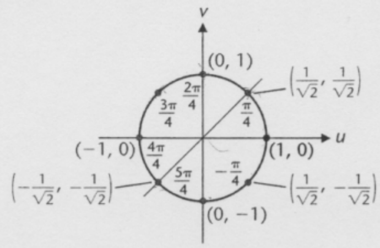
(A) $W(5\pi/4)$ (B) $W(-\pi/4)$

The student was interrupted at this point and asked to work this example without looking at the solution which was covered with a Post-it© note. The student was then asked to read the solution and work Matched Problem 2.

Solutions (A) Starting at (1, 0) and counting in one-eighth circle steps counterclockwise ($\pi/4$, $2\pi/4$, $3\pi/4$, $4\pi/4$, $5\pi/4$), we find ourselves in the third quadrant on the circle halfway between (−1, 0) and (0, −1), as indicated in Figure 6 on the next page. Using symmetry with respect to the origin, we have

$$W\left(\frac{5\pi}{4}\right) = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

FIGURE 6



(B) Starting at (1, 0), we proceed one-eighth the way around the unit circle in a clockwise direction and end up in the fourth quadrant on the circle halfway between (0, −1) and (1, 0), as indicated in Figure 6. Using symmetry with respect to the horizontal axis, we see that

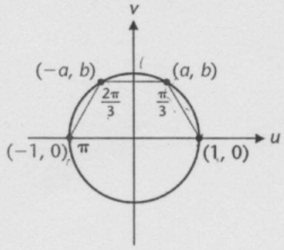
$$W\left(-\frac{\pi}{4}\right) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

MATCHED PROBLEM 2 Find the coordinates of the circular points.

(A) $W(3\pi/4)$ (B) $W(-7\pi/4)$

After checking answers, the student was asked to continue reading:

FIGURE 7
Circular point $W(\pi/3)$.



We continue our investigation by finding the coordinates of the circular point $W(\pi/3)$. Referring to Figure 7, we divide the upper semicircle from (1, 0) to (−1, 0) into thirds. The circular points $W(\pi/3)$ and $W(2\pi/3)$ are symmetric with respect to the v axis; hence, if $W(\pi/3)$ is given coordinates (a, b), then $W(2\pi/3)$ must have coordinates $(-a, b)$. The chord joining $W(2\pi/3)$ and $W(\pi/3)$ is thus $2a$ units long. Using the distance formula (see Section 1-1), we find the length of the chord joining $W(0)$ and $W(\pi/3)$ to be given by $\sqrt{(a-1)^2+b^2}$. The two chords are equal in length, since congruent arcs are opposite congruent chords on the same circle. Thus,

$$\sqrt{(a-1)^2+b^2} = 2a$$

Squaring both sides, we obtain

$$\begin{aligned}
(a-1)^2+b^2 &= 4a^2 \\
a^2-2a+1+b^2 &= 4a^2 \\
a^2+b^2-2a+1 &= 4a^2 \\
1-2a+1 &= 4a^2 \qquad a^2+b^2=1 \text{ (Why?)} \\
4a^2+2a-2 &= 0 \\
2a^2+a-1 &= 0 \\
(2a-1)(a+1) &= 0 \\
a=\tfrac{1}{2} \quad \text{or} \quad a &= -1 \\
a &= \tfrac{1}{2} \qquad a=-1 \text{ must be discarded. (Why?)}
\end{aligned}$$

Substitute $a = \frac{1}{2}$ into $a^2 + b^2 = 1$ and solve for b.

Appendix A (Part II) —Precalculus reading passages (continued)

$$\left(\frac{1}{2}\right)^2 + b^2 = 1$$

$$b^2 = \frac{3}{4}$$

$$b = \pm\frac{\sqrt{3}}{2}$$

$$b = \frac{\sqrt{3}}{2} \qquad b = -\sqrt{3}/2 \text{ must be discarded. (Why?)}$$

Thus,

$$W\left(\frac{\pi}{3}\right) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

Proceeding in a similar manner, or using symmetry with respect to the line $v = u$, we can obtain

$$W\left(\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

The key results from the preceding discussion for the first quadrant are summarized in Figure 8.

COORDINATES OF KEY CIRCULAR POINTS

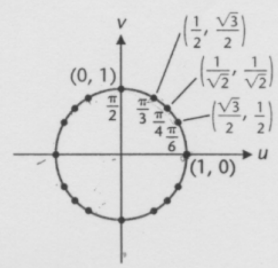


FIGURE 8

It is important that you memorize these first quadrant relationships.

Explore/Discuss 1

An effective **memory aid** for recalling the coordinates of the key circular points in Figure 8 can be created by writing the coordinates of the circular points $W(0)$, $W(\pi/6)$, $W(\pi/4)$, $W(\pi/3)$, and $W(\pi/2)$, keeping this order, in a form where each numerator is the square root of an appropriate number and each denominator is 2. For example, $W(0) = (1, 0) = (\sqrt{4}/2, \sqrt{0}/2)$. Describe the pattern that results.

The student was stopped and was asked if he/she would do the Explore/Discuss. None did. The student was then asked to continue reading:

342 | 5 TRIGONOMETRIC FUNCTIONS

The reason for memorizing the coordinates of key circular points in the first quadrant is that by using these, along with the symmetry of the unit circle, we can find the coordinates of *any* circular point that corresponds to *any* integer multiple of $\pi/6$, $\pi/4$, $\pi/3$, and $\pi/2$.

asked to try without looking at solution

EXAMPLE 3

Finding Coordinates of Circular Points

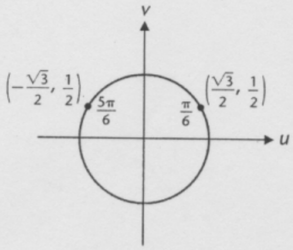
Find the coordinates of the circular points.

(A) $W(5\pi/6)$ (B) $W(-2\pi/3)$

Again, the student was asked to stop and to try to work this example with the solution covered. Then the student read the solution, worked Matched Problem 3, and continued reading.

Solutions

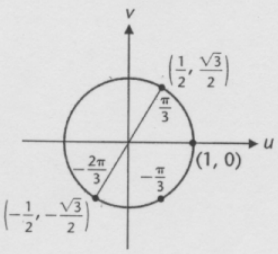
FIGURE 9



(A) Note that $5\pi/6$ is $\pi/6$ less than $\pi = 6\pi/6$. Locate $5\pi/6$ in the second quadrant and use Figure 8 and symmetry with respect to the vertical axis to find $W(5\pi/6)$ (see Fig. 9).

$$W\left(\frac{5\pi}{6}\right) = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

FIGURE 10



(B) Locate $-2\pi/3$ in the third quadrant and use Figure 8 and symmetry with respect to the origin to find $W(-2\pi/3)$ (see Fig. 10).

$$W\left(-\frac{2\pi}{3}\right) = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

MATCHED PROBLEM **3**

Find the coordinates of the circular points.

(A) $W(5\pi/3)$ (B) $W(-7\pi/6)$

The Wrapping Function Is Not One-to-One

It is easy to see that the wrapping function is not a one-to-one function. Each domain value, a real number, corresponds to exactly one range value, a point on the unit circle. However, each range value, a point on the unit circle, corresponds to infinitely many domain values, real numbers. For example, we see that

$$W\left(\frac{\pi}{2}\right) = (0, 1)$$

That is, exactly one range value corresponds to the domain value $\pi/2$. But how many domain values correspond to the range value (0, 1)? Every time we go around the unit circle 2π units in either direction from (0, 1), we return to the same point. Thus, if we are asked to solve

$$W(x) = (0, 1)$$

we have to write

$$x = \frac{\pi}{2} + 2k\pi \qquad k \text{ any integer}$$

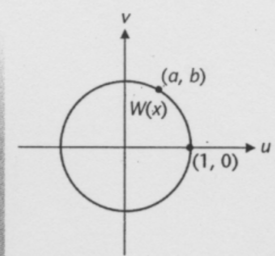
and there are infinitely many domain values of W that correspond to the range value (0, 1). In general, the following theorem applies.

THEOREM 1

A WRAPPING FUNCTION PROPERTY

For all real numbers x,

$$W(x) = W(x + 2k\pi) \qquad k \text{ any integer*}$$



We will have more to say about the implications of this important property of the wrapping function in subsequent sections.

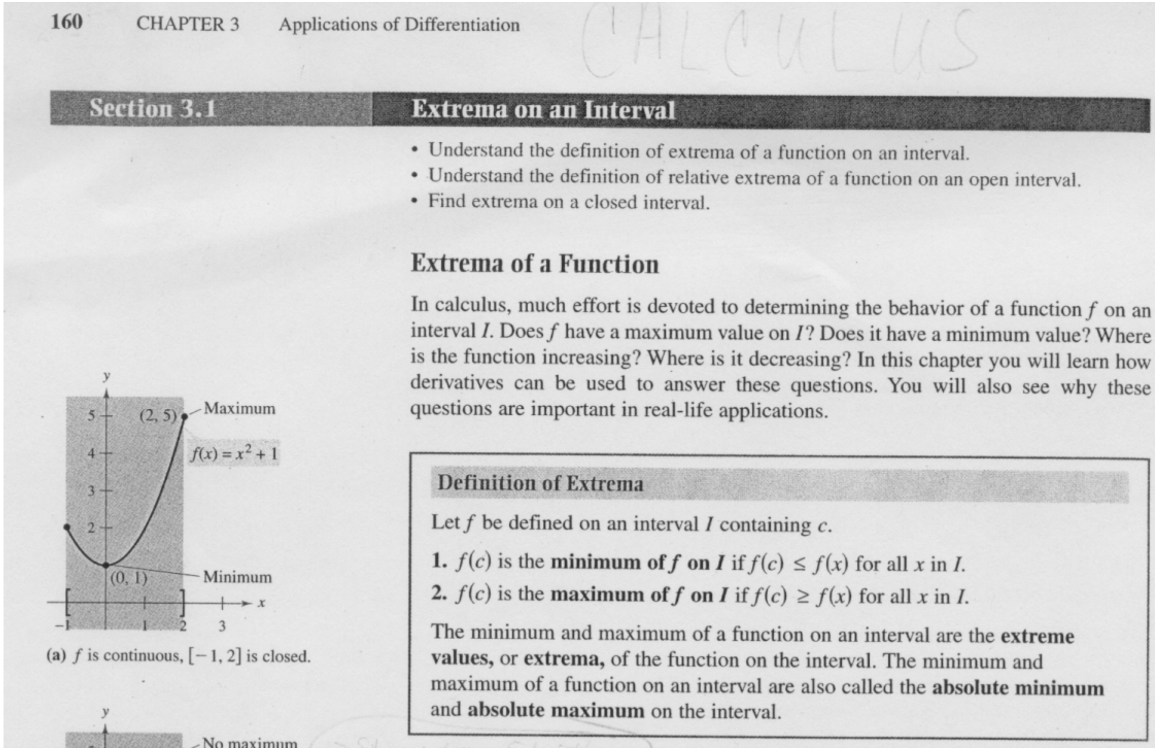
Explore/Discuss 2

(A) Solve the circular point equation $W(x) = (0, -1)$, $-2\pi \le x \le 2\pi$.

(B) Write an expression that would represent all solutions to $W(x) = (0, -1)$.

If the interview got this far, the student was asked to try this Explore/Discuss.

Appendix B Calculus reading passages with interruptions:



Section 3.1 Extrema on an Interval

- Understand the definition of extrema of a function on an interval.
- Understand the definition of relative extrema of a function on an open interval.
- Find extrema on a closed interval.

Extrema of a Function

In calculus, much effort is devoted to determining the behavior of a function f on an interval I. Does f have a maximum value on I? Does it have a minimum value? Where is the function increasing? Where is it decreasing? In this chapter you will learn how derivatives can be used to answer these questions. You will also see why these questions are important in real-life applications.

(a) f is continuous, $[-1, 2]$ is closed.

Definition of Extrema

Let f be defined on an interval I containing c.

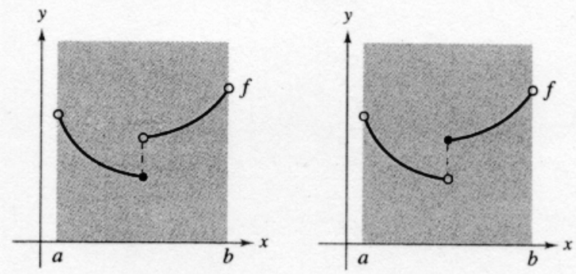
1. $f(c)$ is the **minimum of f on I** if $f(c) \leq f(x)$ for all x in I.
2. $f(c)$ is the **maximum of f on I** if $f(c) \geq f(x)$ for all x in I.

The minimum and maximum of a function on an interval are the **extreme values,** or **extrema,** of the function on the interval. The minimum and maximum of a function on an interval are also called the **absolute minimum** and **absolute maximum** on the interval.

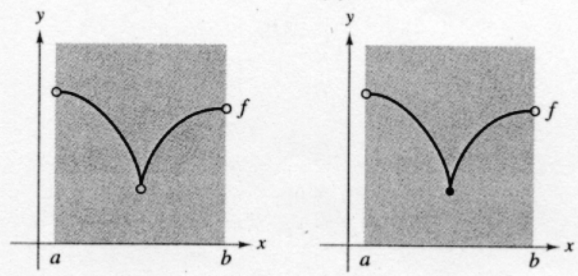
The student was stopped at this point and was asked to try Exercises 51-54 below.

In Exercises 51–54, determine from the graph whether f has a minimum in the open interval (a, b).

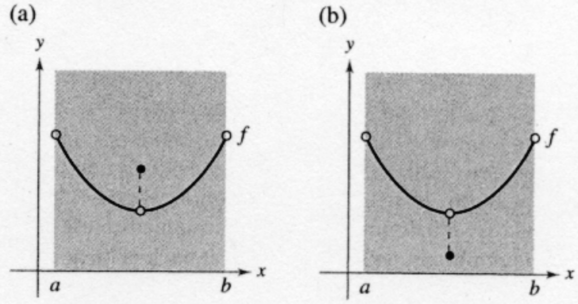
51. (a) (b)



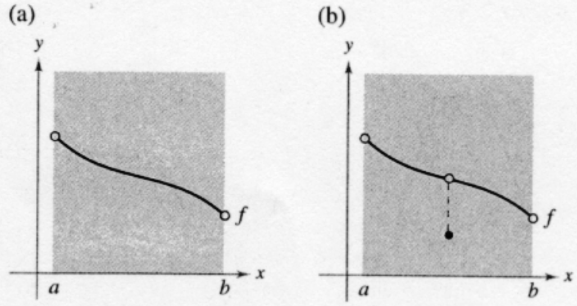
52. (a) (b)



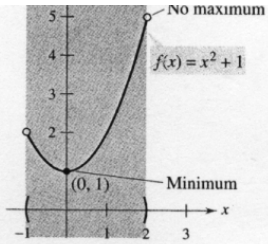
53. (a) (b)



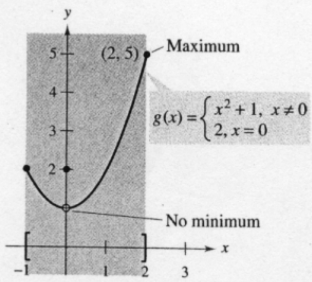
54. (a) (b)



The student was asked to continue reading. (Note some of the figures are on the left side of the first portion of the reading which is on the previous page.)



(b) f is continuous, $(-1, 2)$ is open.



(c) g is not continuous, $[-1, 2]$ is closed.

Extrema can occur at interior points or endpoints of an interval. Extrema that occur at the endpoints are called **endpoint extrema.**
Figure 3.1

A function need not have a minimum or a maximum on an interval. For instance, in Figure 3.1(a) and (b), you can see that the function $f(x) = x^2 + 1$ has both a minimum and a maximum on the closed interval $[-1, 2]$, but does not have a maximum on the open interval $(-1, 2)$. Moreover, in Figure 3.1(c), you can see that continuity (or the lack of it) can affect the existence of an extremum on the interval. This suggests the following theorem. (Although the Extreme Value Theorem is intuitively plausible, a proof of this theorem is not within the scope of this text.)

THEOREM 3.1 The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f has both a minimum and a maximum on the interval.

EXPLORATION

Finding Minimum and Maximum Values The Extreme Value Theorem (like the Intermediate Value Theorem) is an *existence theorem* because it tells of the existence of minimum and maximum values but does not show how to find these values. Use the extreme-value capability of a graphing utility to find the minimum and maximum values of each of the following. In each case, do you think the x-values are exact or approximate? Explain your reasoning.

a. $f(x) = x^2 - 4x + 5$ on the closed interval $[-1, 3]$

b. $f(x) = x^3 - 2x^2 - 3x - 2$ on the closed interval $[-1, 3]$

The student was stopped and was asked if he/she would try the Exploration. Upon completion of the Exploration, student asked to try the two true/false questions below from Calculus textbook, page 167.

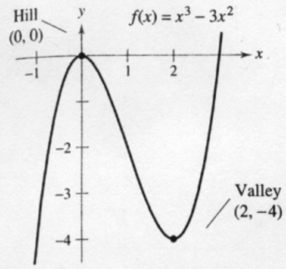
True or False? **In Exercises 61-64, determine whether the statement is true or false.**

If it is false, explain why or give an example that shows it is false.

61. The maximum of a function that is continuous on a closed interval can occur at two different values in the interval.

62. If a function is continuous on a closed interval, then it must have a minimum on the interval.

Then the student was asked to continue reading:



f has a relative maximum at (0, 0) and a relative minimum at (2, −4).
Figure 3.2

Relative Extrema and Critical Numbers

In Figure 3.2, the graph of $f(x) = x^3 - 3x^2$ has a **relative maximum** at the point (0, 0) and a **relative minimum** at the point (2, −4). Informally, you can think of a relative maximum as occurring on a "hill" on the graph, and a relative minimum as occurring in a "valley" on the graph. Such a hill and valley can occur in two ways. If the hill (or valley) is smooth and rounded, the graph has a horizontal tangent line at the high point (or low point). If the hill (or valley) is sharp and peaked, the graph represents a function that is not differentiable at the high point (or low point).

Definition of Relative Extrema

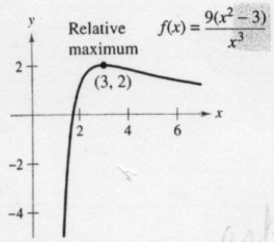
1. If there is an open interval containing c on which $f(c)$ is a maximum, then $f(c)$ is called a **relative maximum** of f.
2. If there is an open interval containing c on which $f(c)$ is a minimum, then $f(c)$ is called a **relative minimum** of f.

The plural of relative maximum is relative maxima, and the plural of relative minimum is relative minima.

Example 1 examines the derivatives of functions at *given* relative extrema. (Much more is said about *finding* the relative extrema of a function in Section 3.3.)

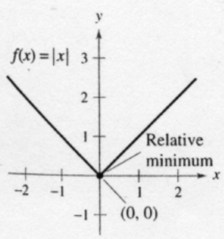
Example 1 The Value of the Derivative at Relative Extrema

Find the value of the derivative at each of the relative extrema shown in Figure 3.3.

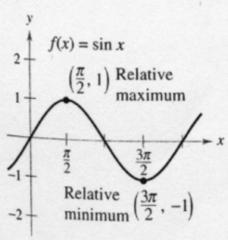


(a) $f'(3) = 0$

The student was stopped and was asked to work Example 1 with the solution covered with Post-It© Notes. Note that the remaining two figures for Figure 3.3 are beside the solution given below. After the student had worked the example, he/she was asked to read the solution and then continue reading.



(b) $f'(0)$ does not exist.



(c) $f\left(\frac{\pi}{2}\right) = 0; f\left(\frac{3\pi}{2}\right) = 0$
Figure 3.3

Solution

a. The derivative of $f(x) = \dfrac{9(x^2 - 3)}{x^3}$ is

$$f'(x) = \frac{x^3(18x) - (9)(x^2 - 3)(3x^2)}{(x^3)^2} \qquad \text{Differentiate using Quotient Rule.}$$

$$= \frac{9(9 - x^2)}{x^4}. \qquad \text{Simplify.}$$

At the point (3, 2), the value of the derivative is $f'(3) = 0$ (see Figure 3.3a).

b. At $x = 0$, the derivative of $f(x) = |x|$ *does not exist* because the following one-sided limits differ (see Figure 3.3b).

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{|x|}{x} = -1 \qquad \text{Limit from the left}$$

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{|x|}{x} = 1 \qquad \text{Limit from the right}$$

c. The derivative of $f(x) = \sin x$ is

$$f'(x) = \cos x.$$

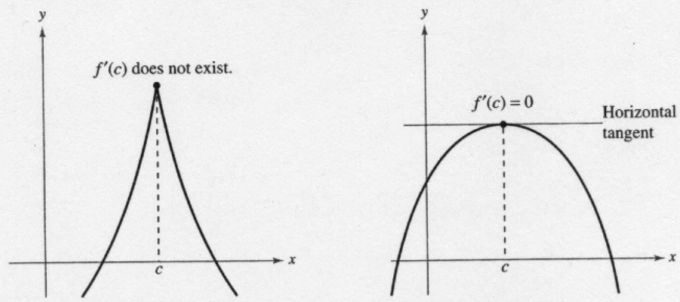
At the point $(\pi/2, 1)$, the value of the derivative is $f'(\pi/2) = \cos(\pi/2) = 0$. At the point $(3\pi/2, -1)$, the value of the derivative is $f'(3\pi/2) = \cos(3\pi/2) = 0$ (see Figure 3.3c).

162 CHAPTER 3 Applications of Differentiation

Note in Example 1 that at the relative extrema, the derivative is either zero or does not exist. The x-values at these special points are called **critical numbers.** Figure 3.4 illustrates the two types of critical numbers.

Definition of a Critical Number

Let f be defined at c. If $f'(c) = 0$ or if f is not differentiable at c, then c is a **critical number** of f.



c is a critical number of f.
Figure 3.4

THEOREM 3.2 Relative Extrema Occur Only at Critical Numbers

If f has a relative minimum or relative maximum at $x = c$, then c is a critical number of f.

PIERRE DE FERMAT (1601–1665)
For Fermat, who was trained as a lawyer, mathematics was more of a hobby than a profession. Nevertheless, Fermat made many contributions to analytic geometry, number theory, calculus, and probability. In letters to friends, he wrote of many of the fundamental ideas of calculus, long before Newton or Leibniz. For instance, the theorem at the right is sometimes attributed to Fermat.

Proof

Case 1: If f is *not* differentiable at $x = c$, then, by definition, c is a critical number of f and the theorem is valid.

Case 2: If f is differentiable at $x = c$, then $f'(c)$ must be positive, negative, or 0. Suppose $f'(c)$ is positive. Then

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} > 0$$

which implies that there exists an interval (a, b) containing c such that

$$\frac{f(x) - f(c)}{x - c} > 0, \text{ for all } x \neq c \text{ in } (a, b). \qquad \text{(See Exercise 58, Section 1.2)}$$

Because this quotient is positive, the signs of the denominator and numerator must agree. This produces the following inequalities for x-values in the interval (a, b).

Left of c: $x < c$ and $f(x) < f(c)$ ⇨ $f(c)$ is not a relative minimum

Right of c: $x > c$ and $f(x) > f(c)$ ⇨ $f(c)$ is not a relative maximum

So, the assumption that $f'(c) > 0$ contradicts the hypothesis that $f(c)$ is a relative extremum. Assuming that $f'(c) < 0$ produces a similar contradiction, you are left with only one possibility—namely, $f'(c) = 0$. So, by definition, c is a critical number of f and the theorem is valid.

At the end of the proof, the student was stopped and was asked what the proof meant to him/her. The student was then asked to continue reading:

SECTION 3.1 Extrema on an Interval 163

Finding Extrema on a Closed Interval

Theorem 3.2 states that the relative extrema of a function can occur *only* at the critical numbers of the function. Knowing this, you can use the following guidelines to find extrema on a closed interval.

Guidelines for Finding Extrema on a Closed Interval

To find the extrema of a continuous function f on a closed interval $[a, b]$, use the following steps.

1. Find the critical numbers of f in (a, b).
2. Evaluate f at each critical number in (a, b).
3. Evaluate f at each endpoint of $[a, b]$.
4. The least of these values is the minimum. The greatest is the maximum.

The next three examples show how to apply these guidelines. Be sure you see that finding the critical numbers of the function is only part of the procedure. Evaluating the function at the critical numbers *and* the endpoints is the other part.

Example 2 **Finding Extrema on a Closed Interval**

Find the extrema of $f(x) = 3x^4 - 4x^3$ on the interval $[-1, 2]$.

The student was stopped and was asked to try the example without looking at the solution which was covered. Then the student was asked to read the solution and continue reading.

Solution Begin by differentiating the function.

$$f(x) = 3x^4 - 4x^3 \qquad \text{Write original function.}$$

$$f'(x) = 12x^3 - 12x^2 \qquad \text{Differentiate.}$$

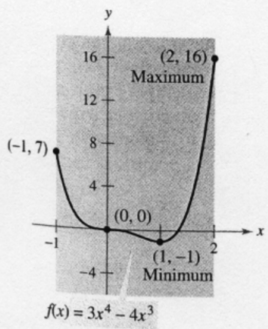
To find the critical numbers of f, you must find all x-values for which $f'(x) = 0$ and all x-values for which $f'(x)$ does not exist.

$$f'(x) = 12x^3 - 12x^2 = 0 \qquad \text{Set } f'(x) \text{ equal to 0.}$$

$$12x^2(x - 1) = 0 \qquad \text{Factor.}$$

$$x = 0, 1 \qquad \text{Critical numbers}$$

Because f' is defined for all x, you can conclude that these are the only critical numbers of f. By evaluating f at these two critical numbers and at the endpoints of $[-1, 2]$, you can determine that the maximum is $f(2) = 16$ and the minimum is $f(1) = -1$, as indicated in the table. The graph of f is shown in Figure 3.5.



$f(x) = 3x^4 - 4x^3$

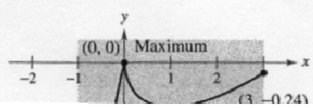
On the closed interval $[-1, 2]$, f has a minimum at $(1, -1)$ and a maximum at $(2, 16)$.

Figure 3.5

Left Endpoint	Critical Number	Critical Number	Right Endpoint
$f(-1) = 7$	$f(0) = 0$	$f(1) = -1$ Minimum	$f(2) = 16$ Maximum

In Figure 3.5, note that the critical number $x = 0$ does not yield a relative minimum or a relative maximum. This tells you that the converse of Theorem 3.2 is not true. In other words, *the critical numbers of a function need not produce relative extrema.*

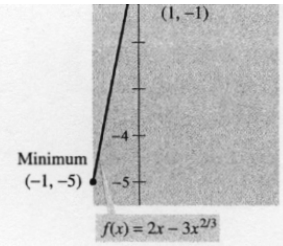
164 CHAPTER 3 Applications of Differentiation



Example 3 **Finding Extrema on a Closed Interval**

Find the extrema of $f(x) = 2x - 3x^{2/3}$ on the interval $[-1, 3]$.

The student was stopped and was asked to try the example without looking at the solution which was covered, and then to read the solution.



On the closed interval $[-1, 3]$, f has a minimum at $(-1, -5)$ and a maximum at $(0, 0)$.
Figure 3.6

Solution Differentiating produces the following.

$$f(x) = 2x - 3x^{2/3} \qquad \text{Write original function.}$$

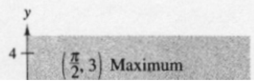
$$f'(x) = 2 - \frac{2}{x^{1/3}} = 2\left(\frac{x^{1/3} - 1}{x^{1/3}}\right) \qquad \text{Differentiate.}$$

From this derivative, you can see that the function has two critical numbers in the interval $[-1, 3]$. The number 1 is a critical number because $f'(1) = 0$, and the number 0 is a critical number because $f'(0)$ does not exist. By evaluating f at these two numbers and at the endpoints of the interval, you can conclude that the minimum is $f(-1) = -5$ and the maximum is $f(0) = 0$, as indicated in the table. The graph of f is shown in Figure 3.6.

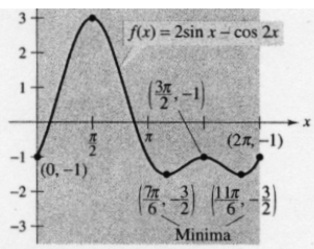
Left Endpoint	Critical Number	Critical Number	Right Endpoint
$f(-1) = -5$ Minimum	$f(0) = 0$ Maximum	$f(1) = -1$	$f(3) = 6 - 3\sqrt[3]{9} \approx -0.24$

Example 4 Finding Extrema on a Closed Interval

Find the extrema of $f(x) = 2 \sin x - \cos 2x$ on the interval $[0, 2\pi]$.



The student was stopped and was asked to try the example without looking at the solution which was covered. The student was then asked to read the solution.



On the closed interval $[0, 2\pi]$, f has two minima at $(7\pi/6, -3/2)$ and $(11\pi/6, -3/2)$ and a maximum at $(\pi/2, 3)$.
Figure 3.7

Solution This function is differentiable for all real x, so you can find all critical numbers by differentiating the function and setting $f'(x)$ equal to zero, as follows.

$$f(x) = 2 \sin x - \cos 2x \qquad \text{Write original function.}$$

$$f'(x) = 2 \cos x + 2 \sin 2x = 0 \qquad \text{Set } f'(x) \text{ equal to 0.}$$

$$2 \cos x + 4 \cos x \sin x = 0 \qquad \sin 2x = 2 \cos x \sin x$$

$$2(\cos x)(1 + 2 \sin x) = 0 \qquad \text{Factor.}$$

In the interval $[0, 2\pi]$, the factor $\cos x$ is zero when $x = \pi/2$ and when $x = 3\pi/2$. The factor $(1 + 2 \sin x)$ is zero when $x = 7\pi/6$ and when $x = 11\pi/6$. By evaluating f at these four critical numbers and at the endpoints of the interval, you can conclude that the maximum is $f(\pi/2) = 3$ and the minimum occurs at *two* points, $f(7\pi/6) = -3/2$ and $f(11\pi/6) = -3/2$, as indicated in the table. The graph is shown in Figure 3.7.

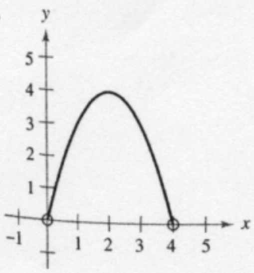
Left Endpoint	Critical Number	Critical Number	Critical Number	Critical Number	Right Endpoint
$f(0) = -1$	$f\left(\frac{\pi}{2}\right) = 3$ Maximum	$f\left(\frac{7\pi}{6}\right) = -\frac{3}{2}$ Minimum	$f\left(\frac{3\pi}{2}\right) = -1$	$f\left(\frac{11\pi}{6}\right) = -\frac{3}{2}$ Minimum	$f(2\pi) = -1$

indicates that in the Interactive 3.0 *CD-ROM and* Internet 3.0 *versions of this text (available at* college.hmco.com*) you will find an Open Exploration, which further explores this example using the computer algebra systems* Maple, Mathcad, Mathematica, *and* Derive.

The student was then stopped (this was the end of the reading section) and asked to try Exercises 8 and 12.

In Exercises 7–10, approximate the critical numbers of the function shown in the graph. Determine whether the function has a relative maximum, relative minimum, absolute maximum, absolute minimum, or none of these at each critical number on the interval shown.

7.



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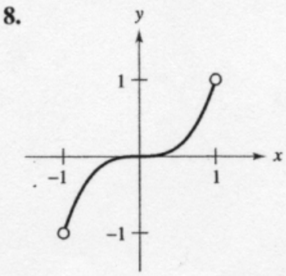

In Exercises 11-16, find any critical numbers of the function.

12. $g(x) = x^2(x^2 - 4)$

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Appendix C—Debriefing questions:

1. Were there any words or terms that bothered you as you read?
2. Were there any symbols or notation that bothered you as you read?
3. Are there any other ways this passage was difficult for you to read and/or understand?
4. What things do you do when you read the textbook?
5. Have you seen the material this passage covered anywhere before? (If so, where?)
6. Did the reading help you do the task? In what way?
7. Is there anything else you would like to say?