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SOME APPLICATIONS OF GRÖBNER BASES IN ROBOTICS AND ENGINEERING

DR. RAFAL ABLAMOWICZ

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TENNESSEE TECHNOLOGICAL UNIVERSITY
Cookeville, TN 38505

Some Applications of Gröbner Bases in Robotics and Engineering

Rafał Abłamowicz

Abstract Gröbner bases in polynomial rings have numerous applications in geometry, applied mathematics, and engineering. We show a few applications of Gröbner bases in robotics, formulated in the language of Clifford algebras, and in engineering to the theory of curves, including Fermat and Bézier cubics, and interpolation functions used in finite element theory.[1]
Keywords: Bézier cubic, Clifford algebra, elimination ideal, envelope, Fermat curve, Gröbner basis, ideal of relations, interpolation function, left ideal, monomial order, quaternion, Rodrigues matrix, syzygy

1 Introduction

Gröbner bases were introduced in 1965 by B. Buchberger. [5–8] For an excellent exposition on their theory see [10,11,17] while for a basic introduction with applications see [9]. These bases gave rise to development of computer algebra systems like muMath, Maple, Mathematica, Reduce, AXIOM, CoCoCA, Macaulay, etc. Buchberger's Algorithm to compute Gröbner bases has been made more efficient [5] or replaced with another approach [12]. Algorithms to compute Gröbner bases have been implemented, for example, in Maple [20], Singular [15,25], and FGb [12]. For a multitude of applications of Gröbner bases see [8, 14] and, in particular, an online repository [16]. For a recent new application in geodesy, see [4].

Department of Mathematics, Box 5054, Tennessee Technological University, Cookeville, TN 38505, USA

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2 Gröbner basis theory in polynomial rings

We follow presentation and notation from [10]. Let $k[x_1,\dots,x_n]$ be a polynomial ring in n indeterminates over a field k. Let $f_1,\dots,f_s$ be polynomials in $k[x_1,\dots,x_n]$. Then $\langle f_1,\dots,f_s\rangle$ denotes an ideal finitely generated by the chosen polynomials. We say that the polynomials form a basis of the ideal. Then, by $\mathbf{V}(f_1,\dots,f_s)$ we denote an **affine variety** defined by $f_1,\dots,f_s$, that is, a subset of k^n, possibly empty, consisting of all common zeros of $f_1,\dots,f_s$, namely

$$\mathbf{V}(f_1,\dots,f_s)=\{(a_1,\dots,a_n)\in k^n \mid f_i(a_1,\dots,a_n)=0 \text{ for all } 1\le i\le s\}.$$

In order to define Gröbner bases in the ideals of $k[x_1,\dots,x_n]$, we first need a concept of a monomial order.

Definition 1. A **monomial order** on $k[x_1,\dots,x_n]$ is any relation $>$ on $\mathbb{Z}^n_{\geq 0}=\{(\alpha_1,\dots,\alpha_n)\mid \alpha_i\in\mathbb{Z}_{\geq 0}\}$, or equivalently, any relation on the set of monomials x^α, $\alpha\in\mathbb{Z}^n_{\geq 0}$, satisfying: (i) $>$ is a total ordering on $\mathbb{Z}^n_{\geq 0}$ (for any $\alpha,\beta\in\mathbb{Z}^n_{\geq 0}$, $\alpha>\beta, \alpha=\beta$ or $\beta>\alpha$); (ii) If $\alpha>\beta$ and $\gamma\in\mathbb{Z}^n_{\geq 0}$, then $\alpha+\gamma>\beta+\gamma$; (iii) $>$ is a well-ordering on $\mathbb{Z}^n_{\geq 0}$ (every nonempty subset has smallest element). We will say that $x^\alpha>x^\beta$ when $\alpha>\beta$. Let $f=\sum_\alpha a_\alpha x^\alpha$ be a nonzero polynomial in $k[x_1,\dots,x_n]$. Then, the **multidegree** of f is

$$\operatorname{multideg}(f)=\max\{\alpha\in\mathbb{Z}^n_{\geq 0} : a_\alpha\neq 0\}$$

where the maximum is taken with respect to $>$. The **leading coefficient** of f is $\mathrm{LC}(f)=a_{\operatorname{multideg}}(f)$; the **leading monomial** of f is $\mathrm{LM}(f)=x^{\operatorname{multideg}}(f)$; and the **leading term** of f is $\mathrm{LT}(f)=\mathrm{LC}(f)\cdot\mathrm{LM}(f)$.

Some of the monomial orders are: (i) lexicographic (lex)[2]: $\alpha>_{lex}\beta$ if, in the vector difference $\alpha-\beta\in\mathbb{Z}^n$, the left-most nonzero entry is positive; (ii) graded reverse lex: $\alpha>_{grevlex}\beta$ if either $|\alpha|>|\beta|$, or $|\alpha|=|\beta|$ and in $\alpha-\beta\in\mathbb{Z}^n$ the right-most nonzero entry is negative; (iii) graded inverse lex: $\alpha>_{ginvlex}\beta$ if either $|\alpha|>|\beta|$, or $|\alpha|=|\beta|$ and in $\alpha-\beta\in\mathbb{Z}^n$ the right-most nonzero entry is positive.

In order to divide any polynomial f by a list of polynomials $f_1,\dots,f_s$, we need a generalized division algorithm.

Theorem 1 (General Division Algorithm). *Fix a monomial order $>$ on $\mathbb{Z}^n_{\geq 0}$, and let $F=(f_1,\dots,f_s)$ be an ordered s-tuple of polynomials. Then every $f\in k[x_1,\dots,x_n]$ can be written as*

$$f=a_1f_1+\cdots a_sf_s+r, \tag{1}$$

where $a_i,r\in k[x_1,\dots,x_n]$ and either $r=0$ or r is a linear combination, with coefficients in k, of monomials, none of which is divisible by any of $\mathrm{LT}(f_1),\dots,\mathrm{LT}(f_s)$. *We call r a* **remainder** *of f on division by F. Furthermore, if $a_if_i\neq 0$, then we have* $\operatorname{multideg}(f)\geq\operatorname{multideg}(a_if_i)$.

[2] In the following, $lex(x_1,\dots,x_n)$ will denote the lex order in which $x_1>x_2>\cdots>x_n$.

The remainder r in (1) is not unique as it depends on the order of polynomials in F and on the monomial order. This shortcoming of the Division Algorithm disappears when we divide polynomials by a Gröbner basis.

Definition 2. Let $I \subset k[x_1,\ldots,x_n]$ be a nonzero ideal. Then, $\mathrm{LT}(I)$ is the set of leading terms of elements of I and $\langle \mathrm{LT}(I)\rangle$ is the ideal generated by the elements of $\mathrm{LT}(I)$.

The ideal $\langle \mathrm{LT}(I)\rangle$ is an example of a monomial ideal. Dickson's Lemma states that every monomial ideal $I \subset k[x_1,\ldots,x_n]$ has a finite basis. [10] Since the polynomial ring $k[x_1,\ldots,x_n]$ is noetherian, the famous theorem of Hilbert states that every ideal $I \subset k[x_1,\ldots,x_n]$ is finitely generated.

Theorem 2 (Hilbert Basis Theorem). *Every ideal $I \subset k[x_1,\ldots,x_n]$ has a finite generating set. That is, $I = \langle g_1,\ldots,g_s\rangle$ for some $g_1,\ldots,g_s \in I$.*

Definition 3. Fix a monomial order. A finite subset $G = \{g_1,\ldots,g_t\}$ of an ideal I is said to be a **Gröbner basis** if $\langle \mathrm{LT}(g_1),\ldots,\mathrm{LT}(g_t)\rangle = \langle \mathrm{LT}(I)\rangle$.

As a consequence of Hilbert's theorem, every ideal $I \subset k[x_1,\ldots,x_n]$ other than $\{0\}$ has a Gröbner basis once a monomial order has been chosen.

When dividing f by a Gröbner basis, we denote the remainder as $r = \overline{f}^G$. Due to the uniqueness of r, one gets unique coset representatives for elements in the quotient ring $k[x_1,\ldots,x_n]/I$: The coset representative of $[f] \in k[x_1,\ldots,x_n]/I$ will be $\overline{f}^G$.

Gröbner bases are computed using various algorithms. The most famous one is the Buchberger's algorithm that uses S-polynomials. One of its many modifications is discussed in [10] whereas [12] implements a completely different approach.

Definition 4. The S-**polynomial** of $f_1, f_2 \in k[x_1,\ldots,x_n]$ is defined as $S(f_1,f_2) = \frac{x^\gamma}{\mathrm{LT}(f_1)}f_1 - \frac{x^\gamma}{\mathrm{LT}(f_2)}f_2$, where $x^\gamma = \mathrm{lcm}(\mathrm{LM}(f_1),\mathrm{LM}(f_2))$ and $\mathrm{LM}(f_i)$ is the leading monomial of f_i w.r.t. some monomial order.[3]

Theorem 3 (Buchberger Theorem). *A basis $\{g_1,\ldots,g_t\} \subset I$ is a Gröbner basis of I if and only if $\overline{S(g_i,g_j)}^G = 0$ for all $i < j$.*

Buchberger's algorithm for finding a Gröbner basis implements the above criterion: If $F = \{f_1,\ldots,f_s\}$ fails because $\overline{S(f_i,f_j)}^G \neq 0$ for some $i < j$, then add this remainder to F and try again.

Gröbner bases computed with the Buchberger's algorithm are usually too large: A standard way to reduce them is to replace any polynomial f_i with its remainder on division by $\{f_1,\ldots,f_{i-1},f_{i+1},\ldots,f_t\}$, removing zero remainders, and for polynomials that are left, making their leading coefficient equal to 1. This produces a *reduced Gröbner basis*. For a fixed monomial order, it is well known that any ideal in $k[x_1,\ldots,x_n]$ has a *unique* reduced Gröbner basis. See, for example, [10] and references therein.

[3] Here, $\mathrm{lcm}(\mathrm{LM}(f_1),\mathrm{LM}(f_2))$ denotes the least common multiple of the leading monomials $\mathrm{LM}(f_1)$ and $\mathrm{LM}(f_2)$.

2.1 Examples of using Gröbner bases

There are many problems, in many different areas of mathematics and applied sciences, that can be solved using Gröbner bases. Here we just list a few applied problems:

- Solving systems of polynomial equations, e.g., intersecting surfaces and curves, finding closest point on a curve or on a surface to the given point, Lagrange multiplier problems (especially those with several multipliers), etc. Solutions to these problems are based on the so called Extension Theory. [10]
- Finding equations for equidistant curves and surfaces to curves and surfaces defined in terms of polynomial equations, such as conic sections, Bézier cubics; finding syzygy relations among various sets of polynomials, for example, symmetric polynomials, finite group invariants, interpolating functions, etc. Solutions to these problems are based on the so called Elimination Theory. [10]
- Finding equidistant curves and surfaces as envelopes to appropriate families of curves and surfaces, respectively. [2, 10]
- The implicitization problem, i.e., eliminating parameters and finding implicit forms for curves and surfaces.
- The forward and the inverse kinematic problems in robotics. [7, 10]
- Automatic geometric theorem proving. [7, 8, 10]
- Expressing invariants of a finite group in terms of generating invariants. [10]
- Finding relations between polynomial functions, e.g., interpolation functions (syzygy relations).
- For recent applications in geodesy see [4].
- See also bibliography on Gröbner bases at Johann Radon Institute for Computational and Applied Mathematics (RICAM). [16]

Our first example is an inverse kinematics problem consisting of finding an elbow point of a robot arm on the circle of intersection between two spheres. This problem is elegantly formulated in the language of conformal algebra CGA in [18, 19].

Example 1 (Elbow of a robot arm) *We model CGA as a Clifford algebra of a 5-dimensional real vector space V which is an extension of 3D Euclidean space by an origin-infinity plane. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5\}$ be basis vectors for V which satisfy the following relations in CGA:*

$$\mathbf{e}_i^2 = 1, \quad \mathbf{e}_i \cdot \mathbf{e}_j = \mathbf{e}_j \cdot \mathbf{e}_i = 0, \quad \mathbf{e}_i \cdot \mathbf{e}_4 = \mathbf{e}_i \cdot \mathbf{e}_5 = 0, \quad \mathbf{e}_4^2 = \mathbf{e}_5^2 = 0, \quad \mathbf{e}_4 \cdot \mathbf{e}_5 = -1. \tag{2}$$

for $i, j = 1, 2, 3$, and $i \neq j$.[4] *Euclidean points, spheres, and planes are modeled, respectively, in CGA by the following 5D vectors:*

[4] We identify $\mathbf{e}_4$ with the origin vector e_0 and $\mathbf{e}_5$ with the infinity vector e_∞ from [18]. Thus, CGA is isomorphic to the Clifford algebra $C\ell_{4,1}$. The dot $\cdot$ denotes the inner product in V.

$$P = \mathbf{p} + \tfrac{1}{2}\mathbf{p}^2 e_\infty + e_0, \quad S = \mathbf{s} + \tfrac{1}{2}(\mathbf{s}^2 - r^2)e_\infty + e_0, \quad \pi = \mathbf{n} + d e_\infty \tag{3}$$

where $\mathbf{p}$ *is the 3D point location,* $\mathbf{s}$ *is the 3D sphere center and* r *is the sphere radius*[5], $\mathbf{n}$ *is the 3D unit normal vector of the plane, and* d *is distance of the plane from the origin. In particular, for a sphere we have* $S^2 = r^2$.[6] *Two spheres* S_1 *and* S_2 *intersect in a 3D circle (resp., a single point, or do not intersect) when* $S_1 \cdot S_2 + r_1 r_2 > 0$ *(resp.,* $S_1 \cdot S_2 + r_1 r_2 = 0$*, or otherwise). The circle is represented in CGA by the element* $C = S_1 \wedge S_2$.

It is shown in [18] that when two spheres intersect in a circle, the bivector C *equals the following quantity:*

$$Z = \mathbf{c} \wedge \mathbf{n_c} - \mathbf{n_c} \wedge e_0 - (\mathbf{c} \cdot \mathbf{n_c})E + [(\mathbf{c} \cdot \mathbf{n_c})\mathbf{c} - \tfrac{1}{2}(\mathbf{c}^2 - r^2)\mathbf{n_c}] \wedge e_\infty \tag{4}$$

where $E = e_\infty \wedge e_0$, $\mathbf{c}$ *is the circle center,* r *is its radius, and vector* $\mathbf{n_c}$ *is normal to the plane* π *in 3D containing the circle. We will solve a system of polynomial equations resulting from the condition* $Z = C$ *for the components of* $\mathbf{c}$, $\mathbf{n_c}$ *and for the radius* r *with a Gröbner basis. For example, let*

$$f_1 = 4(x_1 - 1)^2 + 4x_2^2 + 4x_3^2 - 9 \quad \textit{and} \quad f_2 = (x_1 + 1)^2 + x_2^2 + x_3^2 - 4 \tag{5}$$

be in $\mathbb{R}[x_1, x_2, x_3]$*. Then,* $\mathbf{V}(f_1)$ *and* $\mathbf{V}(f_2)$ *are the two spheres viewed as varieties. These two spheres are represented in CGA as these 1-vectors:*

$$S_1 = \mathbf{e}_1 - \tfrac{5}{8}e_\infty + e_0, \quad S_2 = -\mathbf{e}_1 - \tfrac{3}{2}e_\infty + e_0 \tag{6}$$

Since $S_1 \cdot S_2 + r_1 r_2 = \frac{33}{8} > 0$*, the spheres intersect in a circle* C *given as:*

$$C = S_1 \wedge S_2 = -\tfrac{17}{8}\mathbf{e}_1 \wedge e_\infty + 2\mathbf{e}_1 \wedge e_0 - \tfrac{7}{8} e_0 \wedge e_\infty \tag{7}$$

By letting $C = Z$ *and* $\mathbf{n_c} = (n_1, n_2, n_3)$, $\mathbf{c} = (c_1, c_2, c_3)$*, and by equating symbolic coefficients at corresponding Grassmann basis monomials, we obtain the following system of polynomial equations:*

[5] Of course, we can recover the analytic equation of the sphere by setting $(\mathbf{x} - \mathbf{s})^2 = S^2$ where $\mathbf{x}$ is a vector in 3D from the origin to a surface point on the sphere S.

[6] To be precise, $S^2 = r^2 1$ where 1 is the identity element of CGA. When $r = 0$, the sphere becomes a point.

$$
\begin{aligned}
f_1 &= c_1 n_3 - c_3 n_1 = 0,\\
f_2 &= -c_3 n_2 + c_2 n_3 = 0,\\
f_3 &= c_1 n_2 - c_2 n_1 = 0,\\
f_4 &= 8c_3 n_3 + 8c_2 n_2 + 8c_1 n_1 + 7 = 0,\\
f_5 &= -n_1 - 2 = 0,\\
f_6 &= -n_2 = 0,\\
f_7 &= -n_3 = 0,\\
f_8 &= 8c_1 c_2 n_2 + 4c_1^2 n_1 + 17 + 4n_1 r^2 - 4n_1 c_2^2 + 8c_1 c_3 n_3 - 4n_1 c_3^2 = 0,\\
f_9 &= c_2^2 n_2 + 2c_2 c1 n_1 - n_2 c_1^2 + 2c_2 c_3 n_3 + n_2 r^2 - n_2 c_3^2 = 0,\\
f_{10} &= 2c_3 c_2 n_2 + c_3^2 n_3 + 2c_3 c_1 n_1 - n_3 c_1^2 - n_3 c_2^2 + n_3 r^2 = 0
\end{aligned}
\tag{8}
$$

The reduced Gröbner basis for the ideal I generated by the above ten polynomials in lex order with $n_1 > n_2 > n_3 > c_1 > c_2 > c_3 > r$ is

$$\{-495 + 256r^2, c_3, c_2, -7 + 16c_1, n_3, n_2, n_1 + 2\} \tag{9}$$

from which we get, as expected, that $\mathbf{n_c} = (-2,0,0), \mathbf{c} = (0,0,0)$, and $r = \frac{\sqrt{5}}{2}$. In the same way, one can handle the degenerate cases when the spheres just touch at a single point, when one is included in the other, and when they do not intersect.

Our second example is related to the above and shows how to visualize the circle of intersection of two spheres $C = S_1 \cap S_2$ as an intersection of a cylinder and a plane. Often such visualizations simplify the picture and especially when one considers an additional constraint.

Example 2 *Let S_1 and S_2 be the spheres defined by the polynomials f_1 and f_2 given in (5), that is, $S_1 = \mathbf{V}(f_1)$ and $S_2 = \mathbf{V}(f_2)$. Then, a reduced Gröbner basis for the ideal J generated by these two polynomials for the lex order $x > y > z$ is*

$$G = \{256x_2^2 + 256x_3^2 - 495, 16x_1 - 7\} \tag{10}$$

where the first polynomial c gives the cylinder $\mathbf{V}(c)$ and the second of course is the plane $\mathbf{V}(\pi)$. Thus, the circle $C = S_1 \cap S_2 = \mathbf{V}(c) \cap \mathbf{V}(\pi)$ can be visualized in two different ways: As the intersection of the two spheres Fig. 1 or, as the intersection of the cylinder and the plane Fig. 2. By adding an additional constraint consisting, for example, of an additional plane $\mathbf{V}(\pi_2)$ defined by a polynomial $\pi_2 = x_3 - x_1 - \frac{1}{4}$, we can identify two points on the circle C and the plane π_2. To find their coordinates, it is enough to solve the system of polynomial equations $f_1 = 0, f_2 = 0, \pi_2 = 0$. We can employ the Gröbner basis approach once more by computing a reduced basis for the ideal J generated by f_1, f_2, π_2 for the lex order $x_1 > x_2 > x_3$ and we get

$$G = \{16x_3 - 11, 128x_2^2 - 187, 16x_1 - 7\} \tag{11}$$

which gives the two points $P_1, P_2 = (x_1 = \frac{7}{16}, x_2 = \pm\frac{1}{16}\sqrt{374}, x_3 = \frac{11}{16})$.

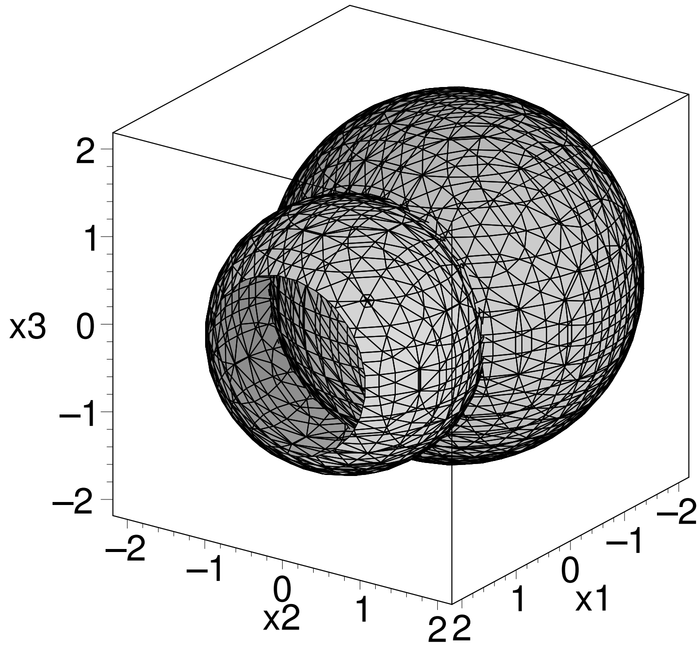

Fig. 1 Circle C as the intersection of two spheres $\mathbf{V}(S_1) \cap \mathbf{V}(S_2)$.

Our third example is a classical problem of finding equidistant curves (envelopes) of various polynomial curves. Here we show how a general envelope of a parabola can be computed in a general case. Such problems also appear in engineering in designing cam mechanisms (cf. [23] and [26]).

Example 3 (Equidistant curves to a parabola) *We compute equidistant curves to a parabola defined by a polynomial*

$$f_1 = 4py_0 - x_0^2 = 0 \tag{12}$$

where $|p|$ *denotes the distance between the focus* $F = (0, p)$ *and the vertex* $V = (0,0)$. *Polynomial* f_2 *defines a circle of radius (offset)* r *centered at a point* (x_0, y_0) *on the parabola* f_1,

$$f_2 = (y - y_0)^2 + (x - x_0)^2 - r^2 = 0 \tag{13}$$

while a polynomial f_3,

$$f_3 = 2xp - 2x_0 p + x_0 y - x_0 y_0 = 0 \tag{14}$$

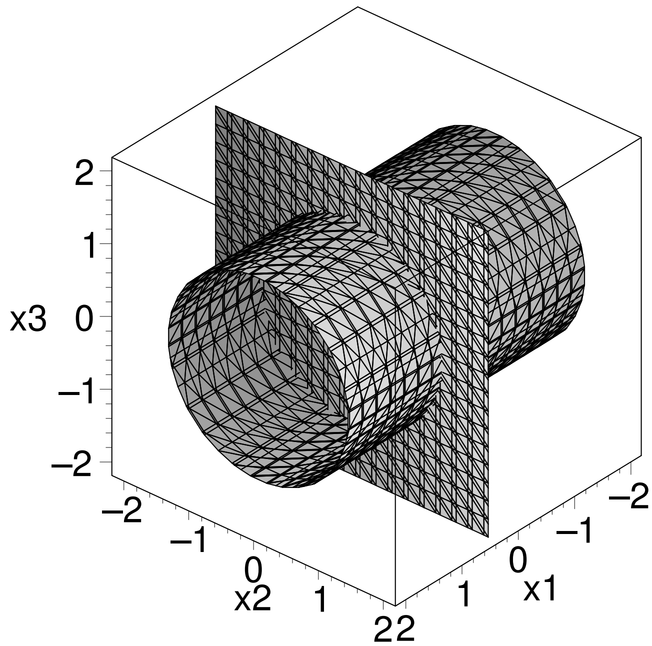

Fig. 2 Circle C as the intersection of the cylinder and the plane $\mathbf{V}(c) \cap \mathbf{V}(\pi_1)$.

gives a condition that a point $P(x,y)$ on the circle f_2 lies on a line perpendicular to the parabola f_1 at the point (x_0,y_0). There are two such points for any given point (x_0,y_0) : one on each side of the parabola. All these points P belong to an affine variety $\mathbf{V} = \mathbf{V}(f_1, f_2, f_3)$ – the envelope *of the family of circles – and define two equidistant curves at the distance r from the parabola. To find a single polynomial equation for this envelope, we compute a reduced Gröbner basis for the ideal $I = \langle f_1, f_2, f_3\rangle \subset \mathbb{R}[x_0,y_0,x,y,p,r]$ for a suitable elimination order. Then, eliminating variables x_0 and y_0 gives a single polynomial $g \in \mathbb{R}[x,y,p,r]$ that defines the envelope. Polynomial g provides a Gröbner basis for the second elimination ideal $I_2 = I \cap \mathbb{R}[x,y,p,r]$.*

The reduced Gröbner basis G for the ideal I for $lex(y_0,x_0,x,y,r,p)$ order consists of fourteen homogeneous polynomials while $I_2 = \langle g\rangle$ where g is as follows:

$$\begin{aligned} g = {} & -2pr^2yx^2 + 8pr^2y^3 + 8p^2r^2y^2 - 32yp^3r^2 + 16p^4r^2 - 16y^4p^2 + 32y^3p^3 \\ & - 16p^4y^2 + 3r^2x^4 + 8p^2r^4 + 20p^2r^2x^2 - y^2x^4 + 10ypx^4 - x^6 - x^4p^2 \\ & + 8py^3x^2 - 32x^2y^2p^2 + 8x^2yp^3 - 3r^4x^2 + 2r^2x^2y^2 + r^6 - r^4y^2 - 8pr^4y \end{aligned} \tag{15}$$

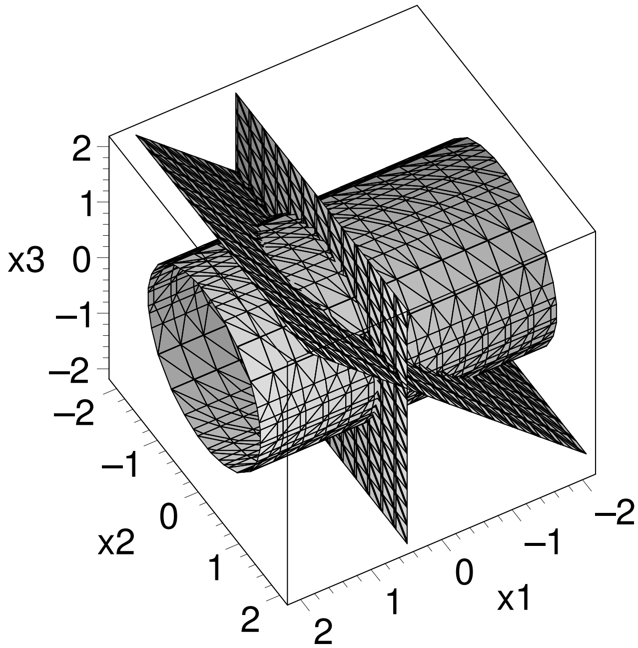

Fig. 3 Two points as the intersection of the circle C and the additional plane $\mathbf{V}(\pi_2)$.

It is possible now to analytically analyze singularities of the envelope by finding points on $\mathbf{V}(g)$ where $\nabla g = 0$. This gives a critical value $r_{crit} = 2|p|$ of r that determines whether $\mathbf{V}(I_2)$ has one or three singular points. For more details, as well as for a complete treatment of other conics, see [2].

In a manner similar to Example 3, it is possible to analyze envelopes and their singularities of other curves defined via polynomial equations like Fermat curves, Bézier cubics, etc., and surfaces, like quadrics, Bézier surfaces. This aids in studying the so called *caustics* [3], shell structures through the finite element analysis [2], and in designing machinery [23].

3 Fermat curves and Bézier cubics

In this section we briefly discuss other curves such as the Fermat curves and the Bézier cubics. We begin with the Fermat curves.

3.1 Fermat curves

The Fermat curves are defined as

$$f_1 = x_0^n + y_0^n - c^n, \quad n \in \mathbb{Z}^+, c > 0.$$

Define f_2 to be the circle of radius r centered at (x_0, y_0) on f_1 and let f_3 give a normal line to f_1 at (x_0, y_0).

Let $n = 3$ and $I = \langle f_1, f_2, f_3 \rangle \subset \mathbb{R}[x_0, y_0, x, y, r, c]$ where

$$f_1 = x_0^3 + y_0^3 - c^3,\ f_2 = (x - x_0)^2 + (y - y_0)^2 - r^2,\ f_3 = xy_0^2 - x_0y_0^2 - x_0^2y + x_0^2y_0$$

For the elimination order *lexdeg*$([x_0, y_0], [x, y, r, c])$,[7] the reduced Gröbner basis for I consists of 129 polynomials of which only one $g \in \mathbb{R}[x, y, r, c]$. Polynomial g has 266 terms and is homogeneous of degree 18. The variety $\mathbf{V}(I)$ contains our offset curves shown in Figure 4.

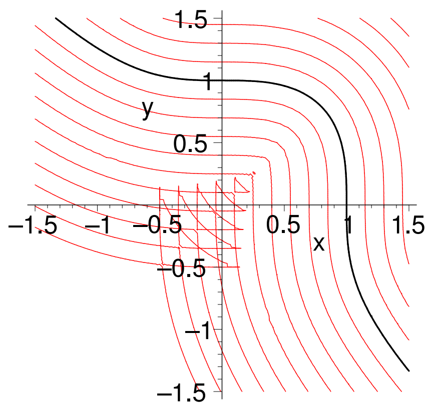

Fig. 4 Fermat cubic $n = 3$ with equidistant curves and growing singularities

Let $n = 4$ and $I = \langle f_1, f_2, f_3 \rangle \subset \mathbb{R}[x_0, y_0, x, y, r, c]$ where

$$f_1 = x_0^4 + y_0^4 - c^4,\ f_2 = (x - x_0)^2 + (y - y_0)^2 - r^2,\ f_3 = xy_0^3 - x_0y_0^3 - x_0^3y + x_0^3y_0$$

[7] In the degree lexicographic order *lexdeg*$([x_0, y_0], [x, y, r, c])$ monomials involving only x_0 and y_0 are compared using the total degree *tdeg*(x_0, y_0) (*tdeg* is also known as the graded reverse lexicographic order *grevlex*); monomials involving only x, y, r, c are compared using the term order *tdeg*(x, y, r, c); a monomial involving x_0 or y_0 is higher than another monomial involving only x, y, r, c. Such a term order is usually used to eliminate the indeterminates listed in the first list namely x_0, y_0 [10].

A reduced Gröbner basis for I for the same order consists of 391 polynomials of which only one $g \in \mathbb{R}[x,y,r,c]$. Polynomial g has 525 terms and is homogeneous of degree 32. The variety $\mathbf{V}(I)$ contains our offset curves shown in Figure 5.

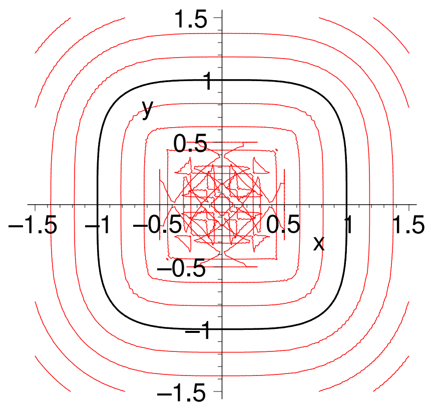

Fig. 5 Fermat cubic $n = 4$ with equidistant curves and growing singularities

3.2 *Bézier cubics*

A Bézier cubic is defined parametrically as:

$$\begin{aligned} X &= (1-t)^3 x_1 + 3t(1-t)^2 x_2 + 3t^2(1-t)x_3 + t^3 x_4, \\ Y &= (1-t)^3 y_1 + 3t(1-t)^2 y_2 + 3t^2(1-t)y_3 + t^3 y_4 \end{aligned} \tag{16}$$

where $(x_i, y_i), i = 1, \ldots, 4$, are the coordinates of four control points, and $0 \le t \le 1$.

3.2.1 First application of Gröbner bases to Bézier cubics

In our first application of Gröbner basis we show how to eliminate the parameter t from the defining polynomials (16) and find a single polynomial that defines the cubic implicitly. We compute a reduced Gröbner basis G for the ideal

$$I = \langle x - X, y - Y \rangle \subset \mathbb{R}[x, y, x_i, y_i, t]$$

for the elimination order $\mathit{lexdeg}([t],[x,y,x_i,y_i])$. The basis G has 12 polynomials of which only one g does not contain t, or, $g \in \mathbb{R}[x,y,x_i,y_i]$. Then, g is homogeneous of degree 6 and it contains 460 terms.

Let the control points be $(\frac{3}{2},0),(0,\frac{1}{4}),(3,2),(2,0)$. Then, the Bézier cubic in Figure 6 has this implicit form:

$$\begin{aligned} g = &-213948x + 66420y - 214164yx - 145656y^2x + 89964x^2y \\ &+ 110079x^2 + 135756 + 78608y^3 - 18522x^3 + 219456y^2 = 0 \end{aligned} \tag{17}$$

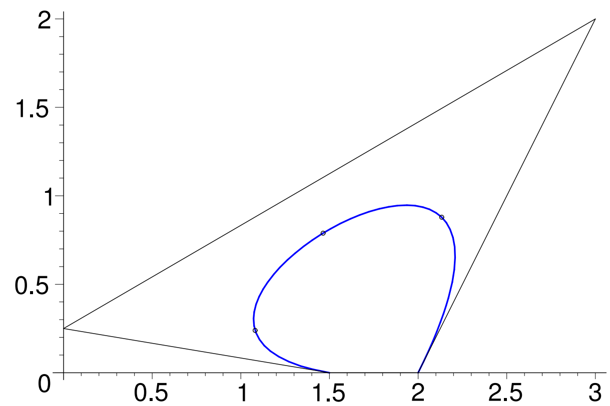

Fig. 6 Bézier cubic in implicit form with control points and points where curvature is maximum or minimum

3.2.2 Second application of Gröbner bases to Bézier cubics

As our second application we find a parameterization $(x(t),y(t))$ for a variety of equidistant curves to a Bézier cubic by computing a reduced Gröbner basis G for a suitable ideal I in elimination order $lexdeg([y],[x,t])$.

Let the control points be $(2,0),(3,3),(4,1),(3,0)$, then

$$X = 2 + 3t - 2t^3, \quad Y = 9t - 15t^2 + 6t^3,$$

and consider an ideal $I = \langle x - X, y - Y, f_3, f_4 \rangle \in \mathbb{R}[x,y,r,t]$ where f_3 gives equation of circle of radius r at $(X(t),Y(t))$ on the cubic

$$f_3 = (y-Y)^2 + (x-X)^2 - r^2, \tag{18}$$

while f_4 gives equation of a normal at $(X(t),Y(t))$ to the cubic

$$\begin{aligned} f_4 = &\, 3x - 6xt^2 - 6 + 417t^2 - 90t - 642t^3 \\ &- 120t^5 + 9y - 30yt + 18yt^2 + 450t^4 \end{aligned} \tag{19}$$

for the offset values $r = \frac{2}{25}, \frac{4}{25}, \frac{6}{25}$. For each r, the basis G has four polynomials $I = \langle h_1, h_2, h_3, h_4 \rangle$: Polynomial h_1 depends only on x, t and is quadratic in x while polynomials h_2, h_3, h_4 depend on x, y, t. The discriminant of h_1 is always positive or zero when $t_s =$ for $0 \leq t \leq 1$. This means that x can always be parameterized in terms of t by solving $h_1 = 0$ for x with radicals [10].

Let gx_{1i}, gx_{2i} be the solutions of $h_1 = 0$ for x for three offset values of r, $i = 1, 2, 3$. Then each gx_{1i}, gx_{2i} is continuous but not smooth. Furthermore, the first elimination ideal is $I_1 = \langle h_1 \rangle = I \cap \mathbb{R}[x,t]$. Here, indices 1 (red) and 2 (blue) refer to opposite sides of the Bézier cubic in Figure 7. The graphs intersect at $t = t_s =$ given above.

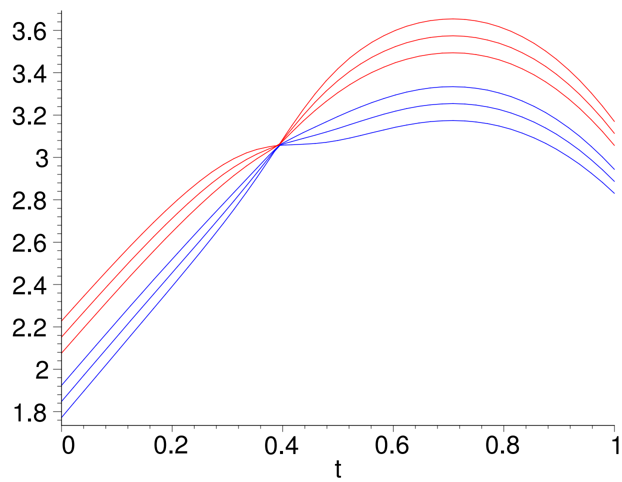

Fig. 7 Graphs of gx_{1i} and gx_{2i} for $i = 1, 2, 3$

Polynomials h_2, h_3 are linear in y while h_4 is quadratic in y, and $h_2, h_3, h_4 \in \mathbb{R}[x,t][y]$. Since one of their leading coefficients is a nonzero constant, by the Extension Theorem [10], any partial solution of $h_1 = 0$ in terms of x and t is extendable to the y-solution.

To find a parameterization for $y = y(t)$, substitute gx_{1i}, gx_{2i} into h_3 for each $i = 1, 2, 3$ and solve for y : We obtain discontinuous functions gy_{1i}, gy_{2i} that belong to the variety $\mathbf{V}(h_2, h_3, h_4)$. The single discontinuity appears at $t = t_s$. See Figure 8.

For the offset values r smaller than r_{crit}, the equidistant curves are smooth curves in the Euclidean plane since, as shown later, they can be defined globally by an equation $g(x, y) = 0$ where g is a polynomial of two variables, and the partial derivatives g_x and g_y are not simultaneously equal to 0. That is, equidistant curves to the Bézier cubic form a non-singular variety when $r < r_{crit}$.

For the Bézier cubic we conjecture that $r_{crit} = \frac{1}{\kappa_{max}} = \rho_{min}$. In this example we chose the three values of r to satisfy this condition. Thus, we can parameterize the equidistant curves for the Bézier cubic in our example as, for one side:

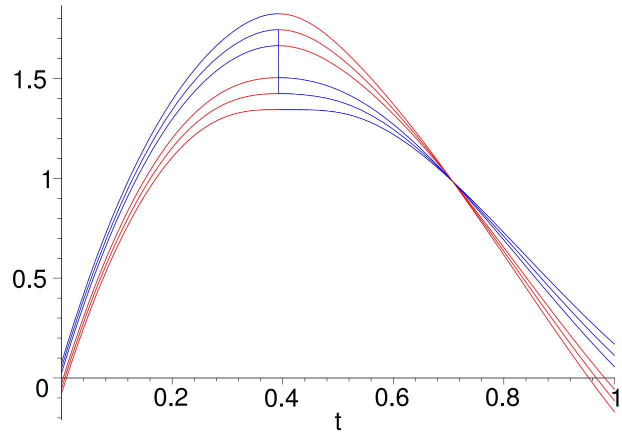

Fig. 8 Graphs of gy_{1i} and gy_{2i} for $i = 1,2,3$

$$Gx_{1i}(t) = \begin{cases} gx_{1i}(t), & t < t_s \\ gx_{2i}(t), & t \geq t_s \end{cases}, \qquad Gy_{1i}(t) = \begin{cases} gy_{1i}(t), & t < t_s \\ y_{1i}, & t = t_s \\ gy_{2i}(t), & t > t_s, \end{cases} \tag{20}$$

where $y_{1i} = \lim_{t \to t_s^-} gy_{1i}(t) = \lim_{t \to t_s^+} gy_{2i}(t)$, $i = 1,2,3$. Likewise for the other side:

$$Gx_{2i}(t) = \begin{cases} gx_{2i}(t), & t < t_s \\ gx_{1i}(t), & t \geq t_s \end{cases}, \qquad Gy_{2i}(t) = \begin{cases} gy_{2i}(t), & t < t_s \\ y_{2i}, & t = t_s \\ gy_{1i}(t), & t > t_s, \end{cases} \tag{21}$$

where $y_{2i} = \lim_{t \to t_s^-} gy_{2i}(t) = \lim_{t \to t_s^+} gy_{1i}(t)$, $i = 1,2,3$.

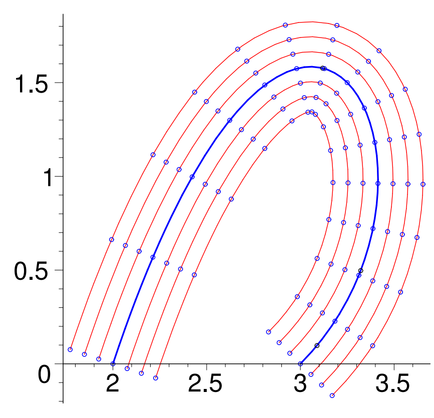

Fig. 9 Bézier cubic with parametric equidistant curves, nodes, points of max/min curvature, and switch points where $t = t_s$

3.2.3 Third application of Gröbner bases to Bézier cubics

In this section we will find a general polynomial $g \in \mathbb{R}[x,y,r]$ so that the polynomial equation $g = 0$ will give equidistant curves to a Bézier cubic at an arbitrary offset r. We first find a reduced Gröbner basis for a suitable ideal in the elimination order $lexdeg([t,X,Y],[x,y,r])$.

Let the control points be $(2,0),(3,3),(4,1),(3,0)$, then define

$$f_1 = X - 2 - 3t + 2t^3, \quad f_2 = Y - 9t + 15t^2 - 6t^3, \tag{22}$$

and consider an ideal $I = \langle f_1, f_2, f_3, f_4 \rangle \subset \mathbb{R}[t,X,Y,x,y,r]$ where f_3 as in (18) gives equation of a circle of radius r at $(X(t),Y(t))$ on the cubic while f_4 gives equation of a normal at $(X(t),Y(t))$ to the cubic

$$f_4 = -x + 2xt^2 + X - 2Xt^2 - 3y + 10yt - 6yt^2 + 3Y \tag{23}$$

for an arbitrary (non-negative) offset r.

The reduced Gröbner basis for I contains twenty seven polynomials, of which only one belongs to $\mathbb{R}[x,y,r]$. That is,

$$I_3 = I \cap \mathbb{R}[x,y,r] = \langle g \rangle$$

where g is of total degree ten in x,y,r and it has 161 terms (but only sixty six terms for any specific value of r.)

In Fig. 10 we plot a few equidistant curves for various offsets with growing singularities of which one is again of the dove-tail type and it appears across the point of maximum curvature.

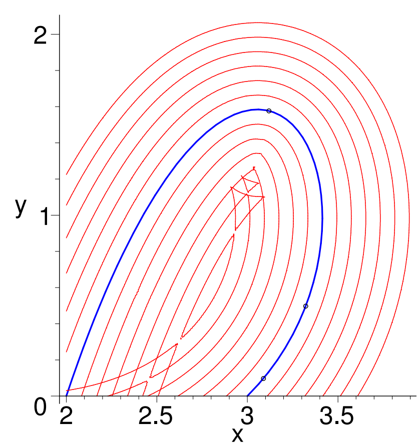

Fig. 10 Bézier cubic with implicit equidistant curves showing growing singularities as the offset r increases.

By analyzing solutions to $\nabla(g)=0$ on $g=0$ for a general offset r, one can search for singularities, if any, of equidistant curves to this specific Bézier cubic.

As an another example, let's define an S-shaped Bézier cubic

$$X = 2-6t+21t^2-\frac{29}{2}t^3, \quad Y = 4-\frac{21}{2}t+18t^2-\frac{23}{2}t^3. \tag{24}$$

Its graph along with a few equidistant offset curves is shown in Figure 11.

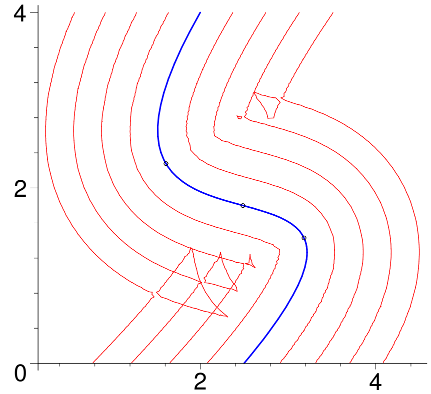

Fig. 11 Bézier cubic with implicit equidistant curves showing growing singularities as the offset r increases.

Let us summarize advantages and disadvantages of this approach.

- **Advantages:**
  - Equidistant curves to any Bézier cubic are given globally as one single polynomial of total degree ten in x,y,r for any offset r.
  - This permits their analytic analysis, including analysis of their singularities, if any, as well as finding the critical value of the offset r_{crit}.
  - Ease of graphing.
  - Ease of finding nodes for finite elements (to any desired accuracy).
- **Disadvantages:**
  - Computational complexity although computation of g for the given cubic, i.e., for the chosen control points, takes only a few seconds.
  - Computation of g for a general Bézier cubic when

    $$I = \langle f_1, f_2, f_3, f_4 \rangle \subset \mathbb{R}[X,Y,x,y,t,r,x_i,y_i], \;\; i=1,2,3,4,$$

    is beyond the computational ability of present-day PC.

Example 4 (Distance to ellipse) *In this example we find a point (or points) on ellipse* $f_1 = \frac{x^2}{a^2}+\frac{y^2}{b^2}-1$ *that minimizes distance from the ellipse to a given point*

$P=(x_0,y_0)$ not on the ellipse and such that $x_0\neq 0$.[8] *Thus, one needs first to find points Q on the ellipse such that a line T tangent to the ellipse at Q is orthogonal to the vector $\overrightarrow{QP}$. Let $f_2=a^2y(x-x_0)-b^2x(y-y_0)$. Then, the condition $f_2=0$ assures that the vector $\overrightarrow{QP}\perp T$. We will use values $x_0=4, y_0=\frac{3}{2}, a=2, b=1$. Thus, we must to solve a system of equations*

$$f_1=4y^2+x^2-4=0 \quad \textit{and} \quad f_2=6yx-32y+3x=0 \tag{25}$$

for x and y. We will find the reduced Gröbner basis for the ideal $I=\langle f_1,f_2\rangle$ that defines $V=\mathbf{V}(f_1,f_2)$ for lex(x,y) order. The basis contains two polynomials

$$\begin{aligned} g_1&=-18y^3+9-9y^2+12x-110y,\\ g_2&=-9-36y+229y^2+36y^3+36y^4 \end{aligned} \tag{26}$$

Observe that g_2 belongs to $I_2=I\cap\mathbb{R}[y]$. Observe also that the leading coefficient in g_1 w.r.t. lex(x,y) is 12, hence by the Extension Theorem [10], every partial solution to the system $\{g_1=0,g_2=0\}$ on the variety $\mathbf{V}(g_2)$ can be extended to a complete solution of (25) on the variety V. Since polynomial g_2 is of degree 4, it's solutions are expressible in radicals. When approximated, two real values of y are $y_1=0.2811025120$ and $y_2=-0.1354474035$. Each of the exact values of y, when substituted into equation $g_1=0$ yields exact value of x. Thus, we have two points Q on the ellipse whose approximate coordinates are $Q_1=(1.919355494, 0.2811025085)$ and $Q_2=(-1.981569077,-0.1354473991)$. Checking the distances, one finds $||\overrightarrow{Q_1P}||=2.411388118<||\overrightarrow{Q_2P}||=6.201117385$, or, that the point Q_1 is closest to the given point P.

Repeating this example in the purely symbolic case when a,b,x_0,y_0 remain unassigned, returns again a two-polynomial reduced Gröbner basis for I :

$$\begin{aligned} G=[&a^4y^4-a^4y^2b^2+2a^2y^2b^4-2a^2b^2y^4+a^2y^2x_0^2b^2+2a^2b^2y^3y_0-\\ &2a^2yb^4y_0-b^6y_0^2-2y^3y_0b^4-y^2b^6+2yb^6y_0+y^4b^4+y^2y_0^2b^4,\\ &a^2b^4y_0-b^6y_0-a^2b^2y^2y_0+b^4y^2y_0+a^4yb^2-2a^2yb^4+yb^6-\\ &\qquad a^2x_0^2yb^2-a^4y^3+2a^2b^2y^3-b^4y^3+x_0b^4xy_0] \end{aligned} \tag{27}$$

where the first polynomial is of degree 4 in y and is, in principle, solvable with radicals. The second polynomial is again of degree 1 in the variable x. Thus, in general, this problem is solvable in radicals.

In [4] a similar problem is studied: It consists of finding distance between a point P on the Earth's topographic surface and the closest to it point p on the *international reference ellipsoid* $\mathbb{E}^2_{a,a,b}$. This is another example of a constrained minimization problem which is set up with the help of a constrained Lagrangian while the resulting system of four polynomial equations is solved with Gröbner basis.

[8] When $x_0=0$ then the solution is obvious.

Example 5 (Rodrigues matrix) *Recall that the trigonometric form of a quaternion $a = a_0 + \mathbf{a} \in \mathbb{H}$ is $a = \|a\|(\cos\alpha + \mathbf{u}\sin\alpha)$, where $\mathbf{u} = \mathbf{a}/|\mathbf{a}|$, $|\mathbf{a}|^2 = {a_1}^2 + {a_2}^2 + {a_3}^2$ and α is determined by $\cos\alpha = a_0/\|a\|$, $\sin\alpha = |\mathbf{a}|/\|a\|$, $0 \leq \alpha < \pi$. Then, any quaternion can be written as*

$$a = \|a\|(\cos\alpha + |\mathbf{a}|^{-1}(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})\sin\alpha). \tag{28}$$

The following theorem can be found in [21, 22].

Theorem 4. *Let a and r be quaternions with non-zero vector parts where $\|a\| = 1$, so $a = \cos\alpha + \mathbf{u}\sin\alpha$ where $\mathbf{u}$ is a unit vector. Then, the norm and the scalar part of the quaternion $r' = ara^{-1}$ equal those of r, that is, $\|r'\| = \|r\|$ and $\mathrm{Re}(r') = \mathrm{Re}(r)$. The vector component $\mathbf{r}' = \mathrm{Im}(r')$ gives a vector $\mathbf{r}' \in \mathbb{R}^3$ resulting from a finite rotation of the vector $\mathbf{r} = \mathrm{Im}(r)$ by the angle 2α counter-clockwise about the axis $\mathbf{u}$ determined by a.*

Let $a = a_0 + \mathbf{a}$, $b = b_0 + \mathbf{b} \in \mathbb{H}$. Let $\mathbf{v}_a$, $\mathbf{v}_b$, and $\mathbf{v}_{ab}$ be vectors in $\mathbb{R}^4$ whose coordinates equal those of $a, b, ab \in \mathbb{H}$.

Then, the vector representation of the product ab is

$$ab \mapsto \mathbf{v}_{ab} = G_1(a)\mathbf{v}_b = G_2(b)\mathbf{v}_a \tag{29}$$

where

$$G_1(a) = \begin{bmatrix} a_0 & -\mathbf{a}^T \\ \mathbf{a} & a_0 I + K(\mathbf{a}) \end{bmatrix}, \quad G_2(b) = \begin{bmatrix} b_0 & -\mathbf{b}^T \\ \mathbf{b} & a_0 I - K(\mathbf{b}) \end{bmatrix}, \tag{30}$$

and

$$K(\mathbf{a}) = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}, \quad K(\mathbf{b}) = \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}, \tag{31}$$

are skew-symmetric matrices determined by the vector parts $\mathbf{a}$ and $\mathbf{b}$ of the quaternions a and b, respectively. For properties of matrices $G_1(a)$ and $G_2(b)$ see [21, 22]. Theorem 4 implies that mapping $r \mapsto r' = ara^{-1}$, $\|a\| = 1$, gives the rotation $\mathbf{r} \mapsto \mathbf{r}'$ in $\mathbb{R}^3$. Using 4×4 matrices, it can be written as:

$$\mathbf{v}_r \mapsto \mathbf{v}'_r = G_1(a)G_2(a^{-1})\mathbf{v}_r = G_1(a){G_2}^T(a)\mathbf{v}_r \tag{32}$$

where

$$G_1(a){G_2}^T(a) = \begin{bmatrix} 1 & 0 \\ 0 & \underbrace{(2{a_0}^2 - 1)I + 2\mathbf{a}\mathbf{a}^T + 2a_0 K(\mathbf{a})}_{R(a)} \end{bmatrix} \tag{33}$$

*The 3×3 matrix $R(a)$ in the product $G_1(a){G_2}^T(a)$ is the well-known **Rodrigues matrix** of rotation. [13] The Rodrigues matrix has this form in terms of the components of a :*

$$R(a)=\begin{bmatrix} a_0^2+a_1^2-a_2^2-a_3^2 & 2a_1a_2-2a_0a_3 & 2a_1a_3+2a_0a_2 \\ 2a_1a_2+2a_0a_3 & a_0^2-a_1^2+a_2^2-a_3^2 & 2a_2a_3-2a_0a_1 \\ 2a_1a_3-2a_0a_2 & 2a_2a_3+2a_0a_1 & a_0^2-a_1^2-a_2^2+a_3^2 \end{bmatrix} \tag{34}$$

Entries of $R(a)$ are homogeneous polynomials of degree 2 in $\mathbb{R}[a_0,a_1,a_2,a_3]$. Separating the scalar and the vector parts of the quaternion r in the $4D$ representation (32), we get

$$\mathrm{Re}(r')=\mathrm{Re}(r), \quad \mathrm{Im}(r')=\boxed{\mathbf{r}'=R(a)\mathbf{r}}=R(a)\,\mathrm{Im}(r) \tag{35}$$

The first relation shows that the scalar part of r remains unchanged, while the vector part $\mathbf{r}'$ of r' is a result of rotation of the vector part $\mathbf{r}$ of r about the axis $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ and the angle of counter-clockwise rotation is 2α. Observe that $\det R(a)=\|a\|^6$ and $R(a)^TR(a)=\|a\|^4I$. Since $\det R(a)=\|a\|^6$ and $R(a)^TR(a)=\|a\|^4I$, the Rodrigues matrix $R(a)$ gives a rotation if and only if $\|a\|=1$. We intend to find the rotation axis $\mathbf{a}$ and the rotation angle 2α by expressing the quaternionic entries (a_0,a_1,a_2,a_3) in terms of the entries of an orthogonal matrix M of determinant 1. For that purpose we will use a technique of Gröbner basis and the theory of elimination. [10] Let $M=(m_{ij})$ be an orthogonal 3×3 matrix, that is, $M^TM=I$. Since M^TM is symmetric, this one constraint gives us six polynomial constraints on the entries of M :

$$c_1=m_{11}^2+m_{21}^2+m_{31}^2-1,\; c_2=m_{12}^2+m_{22}^2+m_{32}^2-1,\; c_3=m_{13}^2+m_{23}^2+m_{33}^2-1,$$
$$c_4=m_{11}m_{12}+m_{21}m_{22}+m_{31}m_{32},\; c_5=m_{11}m_{13}+m_{21}m_{23}+m_{31}m_{33},$$
$$c_6=m_{12}m_{13}+m_{22}m_{23}+m_{32}m_{33}$$

We add one more constraint, namely, that $\det M=1$:

$$c_7=m_{11}m_{22}m_{33}-m_{11}m_{23}m_{32}-m_{21}m_{12}m_{33}+ \\ m_{21}m_{13}m_{32}+m_{31}m_{12}m_{23}-m_{31}m_{13}m_{22}-1$$

A Gröbner basis G_J for the syzygy ideal $J=\langle c_1,c_2,\ldots,c_7\rangle$ with respect to the order lex$(m_{11},m_{12},\ldots,m_{33})$ contains twenty polynomials including five polynomials from the original set. This means that the seven constraint polynomials are not algebraically independent. Define nine polynomials $f_k\in\mathbb{R}[a_0,a_1,a_2,a_3,m_{ij}]$

$$[f_1,f_2,f_3,f_4,f_5,f_6,f_7,f_8,f_9]=[m_{ij}-R(a)_{ij}] \tag{36}$$

Our goal is to express the four parameters a_0,a_1,a_2,a_3 in terms of the nine matrix entries m_{ij} that are subject to the seven constraint (syzygy) relations $c_s=0$, $1\le s\le 7$. This should be possible up to a sign since for any rotation in $\mathbb{R}^3$ given by an orthogonal matrix M, $\det M=1$, there are two unit quaternions a and $-a$ such that $R(a)=R(-a)=M$.

We compute a Gröbner basis G_I for the ideal $I=\langle f_1,\ldots,f_9,c_1,\ldots,c_7\rangle$ for lex$(a_0,a_1,a_2,a_3,m_{11},m_{12},\ldots,m_{33})$ order. G_I contains fifty polynomials of which *twenty polynomials are in* $\mathbb{R}[m_{ij}]$ *: thus they provide a basis* G_J *for the syzygy ideal* J*. We need to solve the remaining thirty polynomial equations for* a_0,a_1,a_2,a_3*, so we divide them into a set* S_l *of twenty linear polynomials in* a_0,a_1,a_2,a_3*, and a set* S_{nl} *of ten non linear polynomials in* a_0,a_1,a_2,a_3*. The first four polynomials in* S_{nl} *are:*

$$\begin{aligned} {a_0}^2 &= \frac{1}{4}(1+m_{11}+m_{22}+m_{33}), \qquad & {a_1}^2 &= \frac{1}{4}(1+m_{11}-m_{22}-m_{33}),\\ {a_2}^2 &= \frac{1}{4}(1-m_{11}+m_{22}-m_{33}), \qquad & {a_3}^2 &= \frac{1}{4}(1-m_{11}-m_{22}+m_{33}), \end{aligned} \tag{37}$$

which easily shows that $\|a\|=1$*, the quaternion a defined by the orthogonal matrix M is a unit quaternion.*

The remaining six polynomials in S_{nl} *are:*

$$\begin{aligned} a_0a_1 &= \frac{1}{4}(m_{32}-m_{23}), \quad & a_0a_2 &= \frac{1}{4}(m_{13}-m_{31}), \quad & a_1a_2 &= \frac{1}{4}(m_{12}+m_{21}),\\ a_0a_3 &= \frac{1}{4}(m_{21}-m_{12}), \quad & a_1a_3 &= \frac{1}{4}(m_{13}+m_{31}), \quad & a_2a_3 &= \frac{1}{4}(m_{23}+m_{32}), \end{aligned} \tag{38}$$

The remaining twenty polynomials from S_l *are linear in* a_0,a_1,a_2,a_3*. Let A be the coefficient matrix of that linear homogeneous system. Matrix A is* 20×4 *but it can be easily reduced to* 14×4 *by analyzing its submatrices and normal forms of their determinants modulo the Gröbner basis* G_J*. It can be shown that this symbolic matrix is of rank* 3*. That is, there is always a one-parameter family of solutions. Once that one-parameter family of solutions is found, two unit quaternions* $\pm a$ *such that* $R(\pm a)=M$ *can be found from remaining ten nonlinear equations.*
For example, let

$$M=\begin{bmatrix} 1 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0 \end{bmatrix}.$$

Then, the linear system $A(a_0,a_1,a_2,a_3)^T=0$ *has the one-parameter solution* $a_0=a_0, a_1=a_0, a_2=0, a_3=0$*. The system of nonlinear equations reduces to just* $4{a_0}^2=2$ *which gives* $a_0=\pm\frac{1}{2}\sqrt{2}$*, and one unit quaternion is:* $a=\frac{1}{2}\sqrt{2}+\frac{1}{2}\sqrt{2}\mathbf{i}=a_0+\mathbf{a}$, $\cos\alpha=\frac{1}{2}\sqrt{2}$, $\sin\alpha=|\mathbf{a}|=\frac{1}{2}\sqrt{2}$*. The Rodrigues matrix gives* $R(\pm a)=M$, $\alpha=\frac{1}{4}\pi$*, so the rotation angle is* $2\alpha=\frac{1}{2}\pi$*, and the rotation axis* $\mathbf{u}$ *is just* $\mathbf{i}$*, as expected.*

For another example, consider the following orthogonal matrix:

$$M=\begin{bmatrix} 0 & \frac{\sqrt{210}-5\sqrt{14}}{35} & \frac{-2\sqrt{35}-5\sqrt{21}}{35}\\ \frac{\sqrt{210}+5\sqrt{14}}{35} & \frac{11}{35} & \frac{-7\sqrt{6}+5\sqrt{10}}{35}\\ \frac{-2\sqrt{35}+5\sqrt{21}}{35} & \frac{-7\sqrt{6}-5\sqrt{10}}{35} & \frac{4}{35} \end{bmatrix}$$

with $\det M=1$*. Then, solution to the linear system is* $a_0=a_0$, $a_1=\frac{-\sqrt{10}}{5}a_0$, $a_2=\frac{-\sqrt{21}}{5}a_0$, $a_3=\frac{\sqrt{14}}{5}a_0$*. Upon substitution into the non-linear equations we find* $a_0=$

$\pm\frac{\sqrt{70}}{14}$ *which eventually gives* $a = \frac{\sqrt{70}}{14} + (-\frac{\sqrt{7}}{7}\mathbf{i} - \frac{\sqrt{30}}{10}\mathbf{j} + \frac{\sqrt{5}}{5}\mathbf{k})$, $\cos\alpha = \frac{\sqrt{70}}{14}$, $\sin\alpha = |\mathbf{a}| = \frac{3\sqrt{14}}{14}$. *It can be verified again that* $R(\pm a) = M$ *and* $\alpha \approx 0.9302740142$ *rad.*

For our last example, we need the following result from [10][Proposition 3, p. 339] on the so called *ideal of relations* I_F for a set of polynomials $F = (f_1,\ldots,f_m)$. It is usually used to derive the syzygy relations among homogeneous invariants of finite groups. We will use it to show how one can systematically derive relations among interpolation functions used in finite element theory. Although these relations are well known [24], their systematic derivation with the help of Gröbner basis is less known.

Consider the system of equations

$$y_1 = f_1(x_1,...,x_n),\ \ldots,\ y_m = f_m(x_1,...,x_n).$$

Then, the syzygy relations among the polynomials $f_1,\ldots,f_m$ can be obtained by eliminating $x_1,...,x_n$ from these equations. Let $k[x_1,\ldots,x_n]^G$ denote the ring of invariants of a finite group $G \subset \mathrm{GL}(n,k)$. The following result is proven in [10].

Proposition 1 *If* $k[x_1,\ldots,x_n]^G = k[f_1,\ldots,f_m]$, *consider the ideal*

$$J_F = \langle f_1 - y_1,\ldots,f_m - y_m,\rangle \subset k[x_1,\ldots,x_n,y_1,\ldots,y_m].$$

(i) I_F *is the nth elimination ideal of* J_F. *Thus,* $I_F = J_F \cap k[y_1,\ldots,y_m]$. *(ii) Fix a monomial order in* $k[x_1,\ldots,x_n,y_1,\ldots,y_m]$, *where any monomial involving one of* $x_1,\ldots,x_n$ *is greater than all monomials in* $k[y_1,\ldots,y_m]$ *and let GB be a Gröbner basis of* J_F. *Then* $GB \cap k[y_1,\ldots,y_m]$ *is a Gröbner basis for* I_F *in the monomial order induced on* $k[y_1,\ldots,y_m]$.

The ideal I_F is known to be a prime ideal of $k[y_1,\ldots,y_m]$.[9] Furthermore, the Gröbner basis for I_F may not be minimal: This is because the original list F of polynomials may contain polynomials which are algebraically dependent. Thus, in order to obtain a minimal Groebner basis for I_F, or, the smallest number of syzygy relations, one needs to assure first that the list F is independent. [10]

We will use this proposition encoded in a procedure `SyzygyIdeal` from the SP package [1] to compute syzygy relations among interpolation functions for orders $k = 2$ and $k = 3$ for triangular elements in finite element method. These elements are referred to as *quadratic* and *cubic* as their interpolation functions are, respectively, quadratic and cubic polynomials, and they contain, respectively, three and four equally spaced nodes per side. For all definitions see [24, Chapter 9].

Example 6 *We derive relations between the interpolation functions for the higher-order Lagrange family of triangular elements with the help of the so called area coordinates* L_i. *For triangular elements there are three non-dimensional coordinates* L_i, $i = 1,2,3$ *such that*

[9] An ideal $I \subset k[x_1,\ldots,x_n]$ is **prime** if whenever $f,g \in k[x_1,\ldots,x_n]$ and $fg \in k[x_1,\ldots,x_n]$, then either $f \in I$ or $g \in I$.

$$L_i = A_i/A, \qquad A = A_1 + A_2 + A_3, \qquad L_1 + L_2 + L_3 = 1 \tag{39}$$

(see [24, Fig 9.3, p. 408]). Here, A_i is the area of a triangle formed by the nodes j and k (i.e., $i \neq j, i \neq k$) and an arbitrary point P in the element, and A is the total area of the element. Each L_i is a function of the position of the point P. For example, if point P is positioned on a line joining nodes 2 and 3 (or, at the nodes 2 or 3) then $L_1 = 0$ and $L_1 = 1$ when P is at the node 1. Thus, L_1 is the interpolation function ψ_1 associated with the node 1 and likewise for L_2 and L_3, that is,

$$\psi_1 = L_1, \quad \psi_2 = L_2, \quad \psi_3 = L_3$$

for any triangular element. Functions (polynomials) L_i are used to construct interpolation functions for higher-order triangular elements with k nodes per side.

For $k = 3$ we have three equally spaced nodes per side of a triangular element and the total number of nodes in the element is $n = \frac{1}{2}k(k+1) = 6$. Then, the degree d of the interpolation functions (polynomials) L_i is $d = k - 1 = 2$. The triangular elements are then called quadratic.

We define six interpolation functions ψ_s, $s = 1, \ldots, 6$, in terms of L_i as in formulas (9.16a), (9.16b), and (9.16c) in [24, p. 410]:

$$\begin{gathered}
\psi_1 = 2L_1^2 - L_1, \quad \psi_2 = 4L_1L_2, \quad \psi_3 = 2L_2^2 - L_2, \\
\psi_4 = -4L_1L_2 - 4L_2^2 + 4L_2, \\
\psi_5 = 2L_1^2 + 4L_1L_2 - 3L_1 + 2L_2^2 - 3L_2 + 1, \\
\psi_6 = -4L_1^2 - 4L_1L_2 + 4L_1.
\end{gathered} \tag{40}$$

The procedure `SyzygyIdeal` *yields seven polynomial relations between the functions ψ_s:*

$$\begin{gathered}
r_1 = \psi_4^2 + 4\psi_4\psi_5 + 4\psi_5^2 + 2\psi_4\psi_6 + 4\psi_5\psi_6 + \psi_6^2 - \psi_4 - 4\psi_5 - \psi_6, \\
r_2 = 2\psi_2\psi_5 + \psi_2\psi_6 + 2\psi_3\psi_6, \\
r_3 = 4\psi_3\psi_5 + 4\psi_4\psi_5 + 4\psi_5^2 - \psi_2\psi_6 - 2\psi_3\psi_6 + \psi_4\psi_6 + 4\psi_5\psi_6 \\
+\psi_6^2 - 4\psi_5 - \psi_6, \\
r_4 = \psi_1 - 1 + \psi_2 + \psi_3 + \psi_4 + \psi_5 + \psi_6, \\
r_5 = \psi_2\psi_4 - \psi_2\psi_6 - 4\psi_3\psi_6 - \psi_4\psi_6, \\
r_6 = 2\psi_3\psi_4 - 6\psi_4\psi_5 - 8\psi_5^2 + 2\psi_2\psi_6 + 6\psi_3\psi_6 - \psi_4\psi_6 - 8\psi_5\psi_6 - 2\psi_6^2 \\
+8\psi_5 + 2\psi_6, \\
r_7 = \psi_2^2 + 4\psi_2\psi_3 + 4\psi_3^2 + 8\psi_4\psi_5 + 12\psi_5^2 - 2\psi_2\psi_6 - 4\psi_3\psi_6 + 2\psi_4\psi_6 \\
+12\psi_5\psi_6 + 3\psi_6^2 - \psi_2 - 4\psi_3 - 12\psi_5 - 3\psi_6.
\end{gathered} \tag{41}$$

It can be verified by direct substitution of (40) into (41) that the latter well-known relations [24] are satisfied.

When there are $k = 4$ equally spaced nodes per side of a triangular element, then the total number of nodes per element is $n = 10$ and the degree d of the interpolation functions (polynomials) is $d = k - 1 = 3$. In a similar manner as above one can derive twenty-nine relations among the ten interpolation polynomials.

4 Conclusions

Our goal was to show a few applications of Gröbner basis technique to engineering problems extending from robotics through curve theory to finite-element method. While applications to the inverse kinematics are well known, formulation of the problem of finding the circle of two intersecting spheres, that is, the plane of the circle, a normal to the plane, and then the radius and the center of the circle in the language of Clifford (geometric) algebra, gave another opportunity to apply the Gröbner basis technique. Since the method is fast and efficient, it can be employed when solving the elbow problem of the robot arm also when formulated in that language.

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