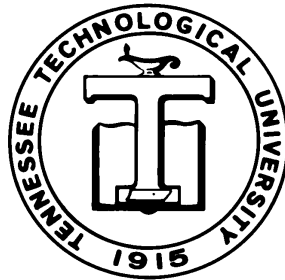

DEPARTMENT OF MATHEMATICS
TECHNICAL REPORT

IDEMPOTENTS OF CLIFFORD ALGEBRAS

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OCTOBER 2003

No. 2003-6



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Idempotents of Clifford Algebras

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Received 09-08-2003

A classification of idempotents in Clifford algebras $C^{p,q}$ is presented. It is shown that using isomorphisms between Clifford algebras $C^{p,q}$ and appropriate matrix rings, it is possible to classify idempotents in any Clifford algebra into continuous families. These families include primitive idempotents used to generate minimal one sided ideals in Clifford algebras. Some low dimensional examples are discussed.

PACS: 02.10.Hh

Key words: Clifford algebra, idempotent, Jordan form, variety

1 Introduction

In higher dimensional quantum field theories as well as in higher dimensional relativistic quantum mechanics, a problem frequently arises how to project onto a specific invariant irreducible subspace of the spinor space. This problem is related to constructing and classifying idempotents of the corresponding Clifford algebras. Discrete families of primitive idempotents have been described by Chevalley [1] and are traditionally used to generate spinor representations of Clifford algebras [2, 3].

In this paper we classify idempotents for arbitrary Clifford algebra $C^{p,q}$ over the reals. Some low dimensional examples are discussed.

In order to find all idempotent elements in a Clifford algebra $C^{p,q}$, we use its matrix representation. Due to the Wedderburn theorem, simple Clifford algebras are isomorphic to full matrix algebras over a division ring. The semisimple Clifford algebras are direct sums of two simple such algebras. In the following table we recall the well known isomorphisms between Clifford algebras $C^{p,q}$ and matrix rings, where $\mathbb{R}, \mathbb{C}, \mathbb{H}$ denote the real numbers, the complex numbers and the quaternions, respectively.

In the following we find all classes of idempotents, up to an isomorphy of the chosen basis, using a specific matrix representation for every particular Clifford algebra $C^{p,q}$.

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Table 1. Isomorphisms of $C^{p,q}$ with matrix rings [4].

when	$p - q \pmod{8} = 3, 7$	then	$C^{p,q} \approx M(2^{\frac{p+q-1}{2}}, \mathbb{C})$
when	$p - q \pmod{8} = 2, 0$	then	$C^{p,q} \approx M(2^{\frac{p+q}{2}}, \mathbb{R})$
when	$p - q \pmod{8} = 4, 6$	then	$C^{p,q} \approx M(2^{\frac{p+q-2}{2}}, \mathbb{H})$
when	$p - q \pmod{8} = 1$	then	$C^{p,q} \approx M(2^{\frac{p+q-1}{2}}, \mathbb{R}) \oplus M(2^{\frac{p+q-1}{2}}, \mathbb{R})$
when	$p - q \pmod{8} = 5$	then	$C^{p,q} \approx M(2^{\frac{p+q-3}{2}}, \mathbb{H}) \oplus M(2^{\frac{p+q-3}{2}}, \mathbb{H})$

2 The case $p - q \pmod{8} = 3, 7$

Let us start with case $C^{p,q} \approx M(2^{\frac{p+q-1}{2}}, \mathbb{C})$. Suppose that $\mathbf{A} \in M(n, \mathbb{C})$ with $N = 2^{\frac{p+q-1}{2}}$ is an idempotent matrix. By means of a similarity transformation it can be transformed to its Jordan form. Consequently, the idempotency implies that the Jordan form of \mathbf{A} must be (up to a transposition of the basis vectors) of the form $\Pi_n = \text{diag}\{\underbrace{1, 1, \dots, 1}_n, \underbrace{0, \dots, 0}_{N-n}\}$, $n = 0, 1, 2, \dots, N$; note two trivial cases:

$n = 0$ ($\mathbf{A} = 0$) and $n = N$ ($\mathbf{A} = I$). Therefore, \mathbf{A} has the form

$$\mathbf{A} = \mathbf{S}\Pi_n\mathbf{S}^{-1} \quad \text{where } \mathbf{S} \in \mathbf{G} = \mathbf{GL}(N, \mathbb{C}).$$

Evidently, the families –i.e. orbits of the $GL(N, \mathbb{C})$ group action– of idempotents can be distinguished by the trace of their elements since these traces are group characters. Each orbit of an idempotent family is generated from a single Π_n . The Π_n form a section of the discrete Weyl group generating the ring of irreducible representations. The stability group of Π_n is the direct product $\mathbf{G}_o := \mathbf{GL}(n, \mathbb{C}) \times \mathbf{GL}(N - n, \mathbb{C})$. The matrices \mathbf{S} can be decomposed as $\mathbf{S}_{\mathbf{G}/\mathbf{G}_o}\mathbf{S}_{\mathbf{G}_o}$ with $\mathbf{S}_{\mathbf{G}_o} \in \mathbf{G}_o$ and $\mathbf{S}_{\mathbf{G}/\mathbf{G}_o} \in \mathbf{G}/\mathbf{G}_o$, $[\mathbf{G}_o, \mathbf{S}] = 0$. It is clear that

$$\mathbf{A} = \mathbf{S}_{\mathbf{G}/\mathbf{G}_o}\Pi_n\mathbf{S}_{\mathbf{G}/\mathbf{G}_o}^{-1},$$

i.e., the manifold of the idempotents in this case forms a variety homeomorphic to the coset space

$$\mathbf{G}/\mathbf{G}_o = \mathbf{GL}(N, \mathbb{C}) / (\mathbf{GL}(n, \mathbb{C}) \times \mathbf{GL}(N - n, \mathbb{C})).$$

Denoting this manifold by $\text{IDEM}_n(C^{p,q})$, we obtain the following families of idempotents for $(p - q) \pmod{8} = 3, 7$:

$$\text{IDEM}_n(C^{p,q}) \approx \mathbf{GL}(N, \mathbb{C}) / (\mathbf{GL}(n, \mathbb{C}) \times \mathbf{GL}(N - n, \mathbb{C})).$$

where $N = 2^{\frac{p+q-1}{2}}$, $n = 0, 1, 2, \dots, N$.

Notice that the real dimension of IDEM_n in these cases is

$$\dim_{\mathbb{R}} \text{IDEM}_n = 2(N^2 - n^2 - (N - n)^2) = 4n(N - n).$$

3 The case $p - q \pmod{8} = 2, 0$

In this case $C^{p,q} \approx M(2^{\frac{p+q}{2}}, \mathbb{R})$. In complete analogy to the previous case we obtain the following classification of idempotents:

$$\text{IDEM}_n(C^{p,q}) \approx \mathbf{GL}(N, \mathbb{R}) / (\mathbf{GL}(n, \mathbb{R}) \times \mathbf{GL}(N - n, \mathbb{R})),$$

where $N = 2^{\frac{p+q}{2}}$, $n = 0, 1, 2, \dots, N$. In particular,

$$\dim_{\mathbb{R}} \text{IDEM}_n = 2n(N - n).$$

4 The case $p - q \pmod{8} = 4, 6$

In this case $C^{p,q} \approx M(2^{\frac{p+q}{2}}, \mathbb{H})$. As before¹⁾ we easily conclude that in this case

$$\text{IDEM}_n(C^{p,q}) \approx \mathbf{GL}(N, \mathbb{H}) / (\mathbf{GL}(n, \mathbb{H}) \times \mathbf{GL}(N - n, \mathbb{H})).$$

$N = 2^{\frac{p+q-2}{2}}$, $n = 0, 2, 4, \dots, N$, and²⁾

$$\dim_{\mathbb{R}} \text{IDEM}_n(C^{p,q}) = 8n(N - n).$$

5 The case $p - q \pmod{8} = 1$

In this case $C^{p,q} = M(2^{\frac{2+q-1}{2}}, \mathbb{R}) \oplus M(2^{\frac{2+q-1}{2}}, \mathbb{R})$. Taking into account idempotency condition we can diagonalize idempotent \mathbf{A} in this case to the form $\Pi_{n,m} = \text{diag}\{\underbrace{1, 1, \dots, 1}_n, \underbrace{0, \dots, 0}_{N-n}; \underbrace{1, 1, \dots, 1}_m, \underbrace{0, \dots, 0}_{N-m}\}$ where $N = 2^{\frac{p+q-1}{2}}$. The stability group of $\Pi_{n,m}$ should be a subgroup of $\mathbf{GL}(N, \mathbb{R}) \times \mathbf{GL}(N, \mathbb{R})$ so it is simply $(\mathbf{GL}(n, \mathbb{R}) \times \mathbf{GL}(N - n, \mathbb{R})) \times (\mathbf{GL}(m, \mathbb{R}) \times \mathbf{GL}(N - m, \mathbb{R}))$. Therefore, the families of idempotents in this case are homeomorphic to the coset space

$$\text{IDEM}_{n,m}(C^{p,q}) \approx \frac{\mathbf{GL}(N, \mathbb{R})}{\mathbf{GL}(n, \mathbb{R}) \times \mathbf{GL}(N - n, \mathbb{R})} \times \frac{\mathbf{GL}(N, \mathbb{R})}{\mathbf{GL}(m, \mathbb{R}) \times \mathbf{GL}(N - m, \mathbb{R})},$$

$N = 2^{\frac{p+q-1}{2}}$, $n = 0, 1, \dots, N$, $m = 0, 1, \dots, N$, and

$$\dim_{\mathbb{R}} \text{IDEM}_{n,m} = 2(n(N - n) + m(N - m)).$$

6 The case $p - q \pmod{8} = 5$

In this case $C^{p,q} = M(2^{\frac{2+q-3}{2}}, \mathbb{H}) \oplus M(2^{\frac{2+q-3}{2}}, \mathbb{H})$. Using (4) we find in analogy to (5) that

$$\text{IDEM}_{n,m}(C^{p,q}) \approx \frac{\mathbf{GL}(N, \mathbb{H})}{\mathbf{GL}(n, \mathbb{H}) \times \mathbf{GL}(N - n, \mathbb{H})} \times \frac{\mathbf{GL}(N, \mathbb{H})}{\mathbf{GL}(m, \mathbb{H}) \times \mathbf{GL}(N - m, \mathbb{H})},$$

¹⁾ The problem of diagonalization of quaternionic matrices and their Jordan form was considered, for example, in [5-7].

²⁾ n is even because of the preservation of the quaternionic structure.

$N = 2^{\frac{p+q-3}{2}}$, $n = 0, 2, 4, \dots, N$, $m = 0, 2, 4, \dots, N$, and

$$\dim_{\mathbb{R}} \text{IDEM}_{n,m} = 8(n(N-n) + m(N-m)).$$

7 Low dimensional examples

7.1 The case $p - q \pmod{8} = 3, 7$

The lowest dimensional examples in this case are $C^{3,0}$ and $C^{1,2}$. We have chosen the following basis for these algebras:

$$\left. \begin{aligned} C^{3,0} &= \text{span}\{I, \vec{\sigma}, i\vec{\sigma}, iI\} \\ C^{1,2} &= \text{span}\{I, \sigma^1, i\sigma^2, i\sigma^3, -i\sigma^1, \sigma^2, -\sigma^3, iI\} \end{aligned} \right\} \sim M(2, \mathbb{C}),$$

where the σ_i 's are Pauli matrices. Thus

$$\mathbf{A} = \frac{1}{2}(I + \vec{\varphi}\vec{\sigma}), \quad \vec{\varphi} = \vec{y} + i\vec{z} \quad \text{where } \vec{y}, \vec{z} \in \mathbb{R}^3.$$

If \mathbf{A} is an idempotent matrix then we obtain two equations for \vec{y}, \vec{z} of the form:

$$\vec{y}^2 - \vec{z}^2 = 1, \tag{1}$$

$$\vec{y}\vec{z} = 0. \tag{2}$$

We see that the hypersurface given by (1) can be identified with the hyperboloid $H_{3,3}$ in \mathbb{R}^6 . It can be shown that (2) represents a rotated cone $V_{3,3}$. Therefore, the variety of idempotents in the Clifford algebras $C^{3,0}$ and $C^{1,2}$ are isomorphic to the intersection of the hyperboloid (1) and the cone (2), $H_{3,3} \cap V_{3,3}$.

$$\text{IDEM}_1(C^{3,0}) = \text{IDEM}_1(C^{1,2}) = \mathbf{GL}(2, \mathbb{C}) / \mathbf{GL}(1, \mathbb{C}) \times \mathbf{GL}(1, \mathbb{C}) \approx H_{3,3} \cap V_{3,3},$$

$$\dim_{\mathbb{R}} \text{IDEM}_1(C^{3,0}) = \dim \text{IDEM}_1(C^{1,2}) = 4 * 1(2-1) = 4.$$

Now we can display exact expressions for the most general idempotent element \mathbf{A} in our examples.

– In $C^{3,0}$, if we denote basis elements as

$$\begin{aligned} e^0 &= I, & e^1 &= \sigma^1, & e^2 &= \sigma^2, & e^3 &= \sigma^3, \\ e^{12} &= i\sigma^1, & e^{23} &= i\sigma^2, & e^{31} &= i\sigma^3, & e^{123} &= iI, \end{aligned}$$

then

$$\mathbf{A} = \frac{1}{2}(e^0 + y_1e^1 + y_2e^2 + y_3e^3 + z_1e^{12} + z_2e^{23} + z_3e^{31}).$$

– In $C^{1,2}$, if we denote basis elements as

$$\begin{aligned} e^0 &= I, & e^1 &= \sigma^1, & e^2 &= i\sigma^2, & e^3 &= i\sigma^3, \\ e^{12} &= -i\sigma^1, & e^{23} &= \sigma^2, & e^{31} &= -\sigma^3, & e^{123} &= iI, \end{aligned}$$

then we find

$$\mathbf{A} = \frac{1}{2}(e^0 + y_1e^1 + z_2e^2 + z_3e^3 - z_1e^{12} + y_2e^{23} - y_3e^{31}).$$

In both cases, \vec{y} and \vec{z} satisfy (1) and (2).

7.2 The case $p - q \pmod{8} = 2, 0$

The lowest dimensional Clifford algebras in this case are $C^{2,0}$ and $C^{1,1}$. We express all elements of these algebras in the following basis:

$$\left. \begin{aligned} C^{2,0} &= \text{span}\{I, \sigma^3, \sigma^1, i\sigma^2\} \\ C^{1,1} &= \text{span}\{I, \sigma^1, i\sigma^2, -\sigma^3\} \end{aligned} \right\} \sim M(2, \mathbb{R}),$$

$$\mathbf{A} = \frac{1}{2}(I + \vec{\varphi}\vec{\sigma}), \quad \vec{\varphi} = (x, iz, y) \quad \text{where } x, y, z \in \mathbb{R},$$

$$\mathbf{A}^2 = \mathbf{A} \Rightarrow \vec{\varphi}^2 = 1,$$

$$\vec{\varphi}^2 = 1 \Rightarrow x^2 + y^2 - z^2 = 1. \tag{3}$$

If \mathbf{A} is an idempotent matrix then we find that $\{\vec{\varphi}^2\}$ defines a one-sheet hyperboloid $H_{2,1}$ given by eq. (3). Therefore the variety of idempotents $\text{IDEM}_1(C^{2,0})$ and $\text{IDEM}_1(C^{1,1})$ is isomorphic to such a hyperboloid.

$$\text{IDEM}_1(C^{2,0}) = \text{IDEM}_1(C^{1,1}) = \mathbf{GL}(2, \mathbb{R})/\mathbf{GL}(1, \mathbb{R}) \times \mathbf{GL}(1, \mathbb{R}) \approx H_{2,1}$$

Note also that $\dim_{\mathbb{R}} \text{IDEM}_1(C^{2,0}) = \dim_{\mathbb{R}} \text{IDEM}_1(C^{1,1}) = 2 * 1(2 - 1) = 2$. We can rewrite the idempotent matrices in the chosen basis for the Clifford algebras under study.

– In $C^{2,0}$ we choose a basis

$$e^0 = I, \quad e^1 = \sigma^3, \quad e^2 = \sigma^1, \quad e^{12} = i\sigma^2,$$

and obtain

$$\mathbf{A} = \frac{1}{2}(e^0 + ye^1 + xe^2 + ze^{12}).$$

– In $C^{1,1}$ we choose a basis

$$e^0 = I, \quad e^1 = \sigma^1, \quad e^2 = i\sigma^2, \quad e^{12} = -\sigma^3,$$

and obtain

$$\mathbf{A} = \frac{1}{2}(e^0 + xe^1 + ze^2 - ye^{12}).$$

In the both cases, x , y , and z satisfy equation (3).

For concrete low dimensional calculations it is convenient to use computer algebra systems. The presented low dimensional examples can be performed using `CLIFFORD3`, a package for Maple⁴) [8, 9].

³) <http://www.math.tntech.edu/rafal/>

⁴) <http://www.maplesoft.com>

8 Conclusions

We have demonstrated that idempotents of Clifford algebras form algebraic varieties. These are smooth manifolds in the natural topology. Since idempotents can be used to generate left and right spinor ideals which carry representations of the algebra under consideration, these varieties contain valuable information about invariant subspaces and the representations. Since Clifford algebras can in general be obtained by tensoring lower dimensional Clifford algebras, present work provides information about tensor product representations that is valuable for branching processes and, possibly, for quantum information processing.

An obvious task for future research is to study intertwiners between the idempotent orbits. This is related to topological questions about the Witt ring on representations and should then be related to Brauer-Wall groups. One might expect to gain insight via, for example, Hopf fibration of spheres into the number and mutual relations of the intertwiners. Furthermore, it may be promising to reinspect the Radon-Hurwitz number involved in the matrix algebra isomorphisms, which counts the number of global vector fields, by performing the present work directly in an algebraic setting without using the matrix representations.

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