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Abstract: This paper addresses the problem of characterizations and constructions of optimal and efficient two-dimensional designs for generalized least squares estimation of treatment contrasts when the errors are correlated. Universally optimal designs are obtained when the plots of each block are on a cylinder and the errors follow the three parameter autonormal process on the cylinder. Efficiencies of planar versions of some of the proposed designs are calculated and found to be very satisfactory.

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1. Introduction

The design problems for correlated errors resulting from neighboring plots which are arranged in rectangular arrays have recently been addressed by several researchers. See Gill and Shukla (1985), Kiefer and Wynn (1981), Kunert (1988), Martin (1982, 1986, 1996), Martin and Eccleston (1993), Morgan and Uddin (1991, 1998), Uddin and Morgan (1991, 1997a, 1997b), and Uddin (1997). In these papers, optimal and highly efficient two dimensional designs have been obtained for certain error covariance structures in conjunction with simplifying assumptions. In addition to simpler models, convenient plot arrangements, and structured error covariance matrices, these authors also introduced different approaches to address the design problems, see Kiefer and Wynn(1981), Martin (1982), and Uddin and Morgan (1997a, 1997b) and compare. Martin (1982) introduced a torus (plots are on a torus) technique and enumerated exact properties of designs on the torus. Following Martin's (1982) torus approach, Morgan and Uddin (1991) considered designs with multiple toruses and described methods for the construction of universally optimal torus designs. Their methods give infinite families of universally optimal torus designs when the errors follow the symmetric second order autonormal process. These torus designs are balanced for nearest (row + column) neighbors and for nearest diagonal neighbors with no like neighbors in any of these directions. The planar versions of these torus designs are shown numerically to have very high efficiency when the errors follow the stationary second order autonormal planar process (see Morgan and Uddin, 1991). An $m_1 \times m_2$ torus block is made an $m_1 \times m_2$ planar one by cutting the torus between any two rows and any two columns. This upsets the row, column and diagonal neighbors for some treatments and hence the loss of perfect neighbor balance and possibly the exact optimal properties in planar applications. Alternatively, one may construct universally optimal designs by taking the plots of each block on a cylinder: an intermediate step between planar and torus processes. With the convention that the rows are circular, a cylindrical block can be made a planar block by cutting the cylinder between any two columns. The planar versions of universally optimal cylinder designs are expected to have better neighbor balance and higher efficiency (due to fewer cuts) than that of the torus designs, although the extent to which this is true depends in part on how the cylinder/torus designs are cut and how the error processes are adapted to the plane. It is this cylindrical approach that is considered in this paper.

The above cylindrical approach is taken in this paper primarily for the purpose of simplicity and feasibility. The algebra involving the generalized least squares information matrix and the maximization of its trace, required for optimality arguments, simplify greatly when blocks are taken on the cylinder. The cylindrical approach is also very attractive from the construction point of view : multiple cylindrical blocks constructed appropriately can be combined together to form new blocks. It will be seen in Section 3 that the procedure leads to designs with reasonable numbers of blocks and treatment replications. It should be noted that the designs with planar blocks are the most relevant one in practice. The present cylindrical setup is closest to the planar case. Furthermore, these cylinder designs can also be laid out as side bordered designs preserving all neighbor properties. Hence one may find these designs useful in other situations, for instance, in polycross experimentations. The reader is referred to Morgan (1990) for a discussion on the usefulness of the proposed designs in situations other than that considered here.

To formulate our problem, let there be b separated blocks each with $m_1 m_2$ experimental units which are arranged on a $m_1 \times m_2$ cylinder where we use the convention that the rows are circular. We address the problem of characterization and construction of cylinder designs for v treatments in $b \quad m_1 \times m_2$ cylindrical blocks that optimally estimate treatment contrasts under the model

$$Y = X\tau + Z\beta + \epsilon, \quad \text{cov}(\epsilon) = \Sigma, \tag{1.1}$$

where Y is the response vector in row-major order, τ is the vector of treatment effects, X is an $n \times v$ $(n = bm_1 m_2)$ plot-treatment design matrix, β is the vector of fixed block effects, and $Z = I_b \otimes J_{m_1 m_2 \times 1}$ is the plot-block incidence matrix. The error covariance matrix considered here is given by the following autonormal process on the cylinder :

$$\sigma^2 \Sigma^{-1} = I_b \otimes (I_{m_1 m_2} - \alpha_1 (I_{m_1} \otimes H_{m_2}) - \alpha_2 (H_{m_1} \otimes I_{m_2}) - \alpha_3 (H_{m_1} \otimes H_{m_2}))$$

where $\alpha_1 > 0, \alpha_2 > 0, \alpha_3 > 0$, and the square matrices H_{m_1} and H_{m_2} of orders m_1 and m_2 are as follows:.

$$(H_{m_1})_{ll'} = \begin{cases} 1, & \text{if } (l - l') = \pm 1, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad (H_{m_2})_{ll'} = \begin{cases} 1, & \text{if } (l - l') = \pm 1 \pmod{m_2}, \\ 0, & \text{otherwise.} \end{cases}$$

For a design d, the generalized least squares information matrix C_d for the estimation of treatment contrasts under (1.1) is

$$C_d = X'\Sigma^{-1}X - X'\Sigma^{-1}Z(Z'\Sigma^{-1}Z)^{-}Z'\Sigma^{-1}X. \tag{1.2}$$

where $(Z'\Sigma^{-1}Z)^{-}$ denotes a generalized inverse of $Z'\Sigma^{-1}Z$. To simplify C_d further, we introduce the following notations:

r_d^δ = diagonal matrix of treatment replications,

N_{dj}^b = $v \times 1$ treatment-block incidence matrix (n_{dij}^b) where n_{dij}^b is the number of times treatment i appears in block j,

N_d^r $= v \times v$ matrix $(n_{dii'}^r)$ where $n_{dii'}^r$ is the number of plots in which treatment i is neighbored by i' in rows,

N_d^c $= v \times v$ matrix $(n_{dii'}^c)$ where $n_{dii'}^c$ is the number of plots in which treatment i is neighbored by i' in columns,

N_d^δ $= v \times v$ matrix $(n_{dii'}^\delta)$ where $n_{dii'}^\delta$ is the number of plots in which treatment i is neighbored by i' in diagonals,

N_{dj}^e $= v \times 1$ incidence matrix (n_{dij}^e) where n_{dij}^e is the replication of treatment i in the two end rows of block j,

Write $w = m_1 m_2(1 - 2\alpha_1 - 2\alpha_2 - 4\alpha_3) + 2m_2(\alpha_2 + 2\alpha_3)$, and $u_j = (1 - 2\alpha_1 - 2\alpha_2 - 4\alpha_3)N_{dj}^b + (\alpha_2 + 2\alpha_3)N_{dj}^e$. Then the matrix C_d given by (1.2) simplifies to

$$C_d = r_d^\delta - \alpha_1 N_d^r - \alpha_2 N_d^c - \alpha_3 N_d^\delta - \frac{1}{w} \sum_{j=1}^{b} u_j u_j'. \tag{1.3}$$

The matrix C_d, for all connected designs d, is positive semidefinite with rank $v - 1$. Note that a design is said to be connected if and only if $\tau_i - \tau_{i'}$ is estimable for all $i \neq i'$. Only connected designs are of interest here, and $D(v, b, m_1, m_2)$ will be used to denote the class of connected designs for the parameters v, b, m_1 and m_2. In section 2, universally optimal designs are determined in some subclasses of $D(v, b, m_1, m_2)$. Constructions of these designs are considered in section 3. Section 4 gives efficiencies of planar versions of some cylinder designs constructed in this paper. Throughout this paper, $BBD(v, b, k)$ is used to refer to a balanced block design (Kiefer, 1975) for v treatments in b blocks each of size k. A $BBD(v, b, k)$ is a $BIBD(v, b, k)$ whenever $k < v$ (Raghavarao, 1971).

2. Optimal Designs.

For the determination of optimal designs in $D(v, b, m_1, m_2)$, we will use the method of Kiefer (1975): a design $d^* \in D(v, b, m_1, m_2)$ which assigns the treatments to the plots in such a way that

(i) $\text{trace}(C_{d^*}) \geq \text{trace}(C_d)$ for all $d \in D(v, b, m_1, m_2)$ and,

(ii) C_{d^*} is completely symmetric,

is universally optimal within the class $D(v, b, m_1, m_2)$.

It follows from (1.3) that trace(C_d), for a design $d \in D(v,b,m_1,m_2)$, is maximized if $n^r_{dii} = n^c_{dii} = n^\delta_{dii} = 0$ for every i, and if $\sum_{i=1}^v \sum_{j=1}^b u_{ij}^2$ is minimized where $u_{ij} = (1-2\alpha_1-2\alpha_2-4\alpha_3)n^b_{dij} + (\alpha_2+2\alpha_3)n^e_{dij}$. This last condition requires that u_{ij}'s are as equal as possible for all i and j, their mean value being $\bar{u} = (1-2\alpha_1-2\alpha_2-4\alpha_3)m_1m_2/v + 2(\alpha_2+2\alpha_3)m_2/v$. In general, it is not immediately clear how these u_{ij}'s can be made as equal as possible. However, this can be done for some special cases.

First we specialize to the case $m_2 \equiv 0 \pmod{v}$. In this case, $u_{ij} = \bar{u}$ if $n^b_{dij} = m_1m_2/v$ and $n^e_{dij} = 2m_2/v$ for all i and j. The conditions for complete symmetry of C_d for a design d having maximal trace(C_d) are $n^r_{dii'}$ equal for $i \neq i'$, $n^c_{dii'}$ equal for $i \neq i'$, and $n^\delta_{dii'}$ equal for $i \neq i'$. Summarizing these we obtain our first optimality result.

Theorem 2.1. *Let $D_1(v,b,m_1,m_2)$ be the class of all connected cylinder designs for v treatments in b $m_1 \times m_2$ blocks, $m_2 \equiv 0$ (mod v). A design $d \in D_1(v,b,m_1,m_2)$ is universally optimal for generalized least squares estimation of treatment contrasts under (1.1) if d satisfies the following conditions:*

(a) $n^r_{dii} = n^c_{dii} = n^\delta_{dii} = 0$ for every i, i.e. no treatment is neighbored by itself in rows, columns, and diagonals,

(b) $n^r_{dii'}$'s are equal for all $i \neq i'$, $n^c_{dii'}$ are equal for all $i \neq i'$, and $n^\delta_{dii'}$ are equal for all $i \neq i'$, i.e. every treatment is neighbored by every other treatment equally often in rows, columns, and diagonals,

(c) $n^b_{dij} = m_1m_2/v$ for all i and j, i.e. each cylindrical block is equireplicate, and

(d) $n^e_{dij} = 2m_2/v$ for all i and j, i.e. for each j, the first and last rows of the j-th block together have each treatment equally often.

Example 1. The following two cylindrical blocks give a universally optimal cylinder design in $D_1(v=5, b=2, m_1=3, m_2=5)$.

$$T_1 = \begin{matrix} 0 & 3 & 1 & 4 & 2 \\ 1 & 4 & 2 & 0 & 3 \\ 2 & 0 & 3 & 1 & 4 \end{matrix}, \quad T_2 = \begin{matrix} 0 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 0 & 1 \\ 4 & 0 & 1 & 2 & 3 \end{matrix}.$$

As mentioned in section 1, the rows of each cylindrical block are circular. This is to say that, in the above seemingly planar displays of T_1 and T_2, the first and the last columns are adjacent columns. This convention will be used henceforth for all cylindrical blocks.

Next we consider the class $D_2(v, b, m_1 = 2, m_2)$ of all connected designs in $2 \times m_2$ cylindrical blocks where we assume $m_2 \not\equiv 0 \pmod{v}$ to avoid overlapping with the class D_1. In this case, there are no interior rows and hence $n_{dij} = n^e_{dij}$ for all i and j, and for all $d \in D_2$. The following theorem follows immediately.

Theorem 2.2. *A design $d \in D_2(v, b, m_1 = 2, m_2)$ is universally optimal for generalized least squares estimation of treatment contrasts under (1.1) if d satisfies the conditions (a) and (b) of Theorem 2.1 and the following condition*

(e) *The b cylindrical blocks of d give a one-dimensional $BBD(v, b, 2m_2)$.*

Example 2. A universally optimal cylinder design in $D_2(v = 6, b = 1, m_1 = 2, m_2 = 15)$ is given by the following cylindrical block:

$$T = \begin{matrix} 2 & 5 & 1 & 3 & 1 & 2 & 4 & 2 & 3 & 5 & 3 & 4 & 1 & 4 & 5 \\ 0 & 3 & 2 & 0 & 4 & 3 & 0 & 5 & 4 & 0 & 1 & 5 & 0 & 2 & 1 \end{matrix}.$$

For $m_2 \not\equiv 0 \pmod{v}$ and $m_1 \geq 3$, we consider the subclass $D_3(v, b, m_1, m_2)$ of designs having equireplicate-blocks (cylindrical). In this case $n^b_{dij} = m_1 m_2 / v$ for all i and j, and for all $d \in D_3$. Then $\sum_{i=1}^{v} \sum_{j=1}^{b} u^2_{ij}$ is minimized if, for each j, $n^e_{dij} = [2m_2/v]$ for $v([2m_2/v]+1) - 2m_2$ treatments and $n^e_{dij} = [2m_2/v]+1$ for the remaining $2m_2 - v[2m_2/v]$ treatments, where [.] stands for the largest integer function. The conditions for complete symmetry of C_d having maximal trace are easily obtained and summarized in the following theorem.

Theorem 2.3. *A design $d \in D_3(v, b, m_1, m_2)$ is universally optimal for generalized least squares estimation of treatment contrasts under the model (1.1) if d satisfies the conditions (a) and (b) of Theorem 2.1 and the following condition :*

(f) *The end-design, i.e. the design obtained by taking the $2m_2$ plots of end-rows of each $m_1 \times m_2$ cylindrical block as a one-dimensional block, is a $BBD(v, b, 2m_2)$.*

Example 3. The following twelve cylindrical blocks give a universally optimal design in $D_3(v = 9, b = 12, m_1 = 3, m_2 = 3)$. Here the end-design is a $BIBD(v = 9, b = 12, k = 6)$.

$$T_1 = \begin{matrix} 0 & 1 & 5 \\ 8 & 7 & 2 \\ 6 & 3 & 4 \end{matrix}, \quad T_2 = \begin{matrix} 8 & 7 & 2 \\ 4 & 6 & 3 \\ 1 & 5 & 0 \end{matrix}, \quad T_3 = \begin{matrix} 4 & 6 & 3 \\ 0 & 1 & 5 \\ 7 & 2 & 8 \end{matrix}, \quad T_4 = \begin{matrix} 0 & 2 & 6 \\ 1 & 8 & 3 \\ 7 & 4 & 5 \end{matrix},$$

$$T_5 = \begin{matrix} 1 & 8 & 3 \\ 5 & 7 & 4 \\ 2 & 6 & 0 \end{matrix}, \quad T_6 = \begin{matrix} 5 & 7 & 4 \\ 0 & 2 & 6 \\ 8 & 3 & 1 \end{matrix}, \quad T_7 = \begin{matrix} 0 & 3 & 7 \\ 2 & 1 & 4 \\ 8 & 5 & 6 \end{matrix}, \quad T_8 = \begin{matrix} 2 & 1 & 4 \\ 6 & 8 & 5 \\ 3 & 7 & 0 \end{matrix},$$

$$T_9 = \begin{matrix} 6 & 8 & 5 \\ 0 & 3 & 7 \\ 1 & 4 & 2 \end{matrix}, \quad T_{10} = \begin{matrix} 0 & 4 & 8 \\ 3 & 2 & 5 \\ 1 & 6 & 7 \end{matrix}, \quad T_{11} = \begin{matrix} 3 & 2 & 5 \\ 7 & 1 & 6 \\ 4 & 8 & 0 \end{matrix}, \quad T_{12} = \begin{matrix} 7 & 1 & 6 \\ 0 & 4 & 8 \\ 2 & 5 & 3 \end{matrix}.$$

3. Constructions Of Optimal and Neighbor Balanced Designs.

The three parameter conditional autonormal error process (planar version of Σ considered under the model (1.1)) is for long-range (slowly decaying) correlations. It is shown elsewhere in the literature that planar designs that are balanced or nearly balanced for neighbors in rows, columns, and diagonals with no like neighbors in any of these directions are highly efficient for such slowly decaying correlations, see Martin (1986), Uddin and Morgan (1991, 1997b), and Morgan and Uddin (1991). All cylinder designs that satisfy the neighbor conditions (a) and (b) have nearly balanced neighbor properties (row, column and diagonal) with no repeated like neighbors when displayed as planar designs and are, therefore, expected to perform very well (see numerical efficiency calculations in section 4) in planar applications. Keeping this in mind, here the construction techniques are developed to give both the universally optimal cylinder designs as well as cylinder designs that satisfy only the neighbor conditions (a) and (b). For convenience, a design that satisfies (a) and (b) will henceforth be referred to as a neighbor balanced cylinder design $(NBCD)$.

We utilize the method of differences (see Raghavarao (1971), for example) on finite fields to obtain $NBCD$s and the universally optimal designs of Section 2. We shall let F_N denote the finite field of order N, and x will be used throughout to denote a primitive root of the field F_N. Our first construction technique is described in the following theorem.

Theorem 3.1. *Let $v = s^n$, where s is an odd prime number and $n \geq 1$. Suppose it is possible to construct m $p \times (q+1)$ planar arrays $A_1, A_2, \ldots, A_m$ of the elements of F_v such that the following conditions are satisfied:*

(i) *the symmetric row-neighbor differences are* $2pqm/(v-1)$ *copies of* $F_v - \{0\}$*,*

(ii) *the symmetric diagonal-neighbor differences are* $4(p-1)qm/(v-1)$ *copies of* $F_v - \{0\}$*,*

(iii) *the symmetric column-neighbor differences obtained from the first* q *columns of all* m *arrays are* $2(p-1)qm/(v-1)$ *copies of* $F_v - \{0\}$*, and*

(iv) *for each* $j = 1, 2, \ldots, m$*, the first column* A_{j1} *and the last column* $A_{j(q+1)}$ *of the array* A_j *satisfy*

$$A_{j(q+1)} - A_{j1} = f_j 1_{p\times 1} \text{ for some } f_j \in F_v - \{0\}.$$

Then there exists an NBCD for $v = s^n$ *treatments in* $b = ms^{n-1}$ *cylindrical blocks each of size* $p \times qs$*. When* $n = 1$*, the design is universally optimal in* $D_1(v = s, b = m, m_1 = p, m_2 = qv)$*.*

Proof: By construction. For each $j = 1, 2, \ldots, m$, define s planar blocks on F_v as follows:

$$B_{jl} = A_j + (l-1)f_j J_{p\times(q+1)},\ l = 1, 2, \ldots, s.$$

Consider first the case $n = 1$. In this case $v = s$, and $(l-1)f_j$ for $l = 1, 2, \ldots, v$ give all elements of F_v. Hence by *(i)* and *(ii)*, the mv arrays B_{jl}'s are balanced for row- and diagonal-neighbors in that every treatment has each other treatment as first neighbor equally often in rows and in diagonals. Also, by *(iii)* the first q columns of all mv arrays together are balanced for column-neighbors. These three conditions also imply that no treatment is neighbored by itself in rows, columns and diagonals in the planar arrays B_{jl}'s. The condition *(iv)* implies that for each j the v arrays $B_{j1}, B_{j2}, \ldots, B_{jv}$ (in this order) are such that the last column of an array is identical to the first column of the next, and the first column of the first array is the same as the last column of the last array. We can then merge these v arrays at their common end columns to form a $p \times qv$ cylindrical array, the last column of B_{jv} is merged with the first column of B_{j1}. Varying $j = 1, 2, \ldots m$, we obtain m cylindrical arrays (blocks). Since the last (or equivalently the first) column of each of the mv arrays B_{jl} is lost in the process of constructing cylinders from planar arrays, the m cylindrical blocks are, by *(iii)*, balanced for column-neighbors. Also, by construction, all row and diagonal neighbors of each treatment in planar arrays are preserved in cylindrical arrays and hence the balanced neighbor properties *(a)* and *(b)* of Theorem 2.1. By construction, each treatment appears exactly q times in each row and hence the equireplication conditions (c) and (d), completing the proof for $n = 1$.

For $n \geq 2$, the s elements $\{(l-1)f_j,\, l = 1, 2, \ldots, s\}$ together give an additive subgroup $F_{(j)} = \{0, f_j, 2f_j, \ldots, (s-1)f_j\}$ of F_v. Hence, similar to the case of $n = 1$, the s blocks $B_{j1}, B_{j2}, \ldots, B_{js}$ can be merged at their common end columns to construct a $p \times qs$ cylindrical block T_j. Using T_j, one may construct s^{n-1} cylindrical blocks as follows:

$$T_{ju} = T_j + h_u^{(j)}\ J_{p\times qs}, u = 1, 2, \ldots, s^{n-1}$$

where $h_1^{(j)} = 0$, and $h_2^{(j)}, h_3^{(j)}, \ldots, h_{s^{n-1}}^{(j)}$ are $s^{n-1} - 1$ distinct elements of F_v each of which belongs to exactly one coset of $F_{(j)}$. Varying $j = 1, 2, \ldots, m$ we obtain ms^{n-1} cylindrical blocks of the desired neighbor balanced cylinder design. To see this it is sufficient to note that the ms^{n-1} cylindrical blocks obtained above use all mv planar blocks $A_j + fJ_{p\times(q+1)}, \forall f \in F_v, j = 1, 2, \ldots, m$; the last column of each of these blocks is lost when merged to construct cylinders. $\square$

The following two examples further illustrate our construction technique.

Example 4. Theorem 3.1 design for $v = 7$ in one 2×21 cylindrical block. Here $m = 1$, and we take

$$A_1 = \begin{matrix} 2 & 4 & 1 & 3 \\ 3 & 6 & 5 & 4 \end{matrix}.$$

The symmetric row, column (exclude the last) and diagonal neighbor differences are each nonzero element of F_7 exactly twice, once, and twice, respectively. Furthermore $f_1 = 1$, thus all conditions of Theorem 1 are satisfied. The seven planar arrays $B_{11}, B_{12}, \ldots, B_{17}$ are

$$B_{11} = \begin{matrix} 2 & 4 & 1 & 3 \\ 3 & 6 & 5 & 4 \end{matrix},\ B_{12} = \begin{matrix} 3 & 5 & 2 & 4 \\ 4 & 0 & 6 & 5 \end{matrix},\ B_{13} = \begin{matrix} 4 & 6 & 3 & 5 \\ 5 & 1 & 0 & 6 \end{matrix},\ B_{14} = \begin{matrix} 5 & 0 & 4 & 6 \\ 6 & 2 & 1 & 0 \end{matrix},$$

$$B_{15} = \begin{matrix} 6 & 1 & 5 & 0 \\ 0 & 3 & 2 & 1 \end{matrix},\ B_{16} = \begin{matrix} 0 & 2 & 6 & 1 \\ 1 & 4 & 3 & 2 \end{matrix},\ B_{17} = \begin{matrix} 1 & 3 & 0 & 2 \\ 2 & 5 & 4 & 3 \end{matrix}.$$

When merged at their common end columns, these arrays yield the following 2×21 cylindrical block which satisfies all conditions of Theorem 2.1.

$$T_1 = \begin{matrix} 2 & 4 & 1 & 3 & 5 & 2 & 4 & 6 & 3 & 5 & 0 & 4 & 6 & 1 & 5 & 0 & 2 & 6 & 1 & 3 & 0 \\ 3 & 6 & 5 & 4 & 0 & 6 & 5 & 1 & 0 & 6 & 2 & 1 & 0 & 3 & 2 & 1 & 4 & 3 & 2 & 5 & 4 \end{matrix}.$$

Example 5. Theorem 3.1 design for $v = 3^2$ in 6 2×12 cylindrical blocks. The nonzero elements of $F_{3^2}(x)$ may be written as $x = (1,0)$, $x^2 = (2,1)$, $x^3 = (2,2)$, $x^4 = (0,2)$, $x^5 = (2,0)$, $x^6 = (1,2)$, $x^7 = (1,1)$, and $x^8 = (0,1)$. Here $m = 2$ and define A_1 and A_2 as follows (where (a,b) is written as (ab) to save space):

$$A_1 = \begin{matrix} (01) & (21) & (02) & (12) & (10) \\ (10) & (22) & (20) & (11) & (22) \end{matrix}, \quad A_2 = \begin{matrix} (10) & (22) & (20) & (11) & (21) \\ (21) & (02) & (12) & (01) & (02) \end{matrix}.$$

Then it may be checked that A_1 and A_2 satisfy the conditions of Theorem 3.1. For these arrays, $f_1 = (12)$, $f_2 = (11)$ and the corresponding additive subgroups are $F_{(1)} = \{(00),(12),(21)\}$, $F_{(2)} = \{(00),(11),(22)\}$. The blocks B_{jl} for $j = 1,2$, $l = 1,2,3$ are

$$B_{11} = \begin{matrix} (01) & (21) & (02) & (12) & (10) \\ (10) & (22) & (20) & (11) & (22) \end{matrix}, \quad B_{12} = \begin{matrix} (10) & (00) & (11) & (21) & (22) \\ (22) & (01) & (02) & (20) & (01) \end{matrix},$$

$$B_{13} = \begin{matrix} (22) & (12) & (20) & (00) & (01) \\ (01) & (10) & (11) & (02) & (10) \end{matrix}, \quad B_{21} = \begin{matrix} (10) & (22) & (20) & (11) & (21) \\ (21) & (02) & (12) & (01) & (02) \end{matrix},$$

$$B_{22} = \begin{matrix} (21) & (00) & (01) & (22) & (02) \\ (02) & (10) & (20) & (12) & (10) \end{matrix}, \quad B_{23} = \begin{matrix} (02) & (11) & (12) & (00) & (10) \\ (10) & (21) & (01) & (20) & (21) \end{matrix}.$$

The blocks B_{11}, B_{12}, and B_{13}, when merged at their common end columns, yield the 2×12 cylindrical block

$$T_1 = \begin{matrix} (01) & (21) & (02) & (12) & (10) & (00) & (11) & (21) & (22) & (12) & (20) & (00) \\ (10) & (22) & (20) & (11) & (22) & (01) & (02) & (20) & (01) & (10) & (11) & (02) \end{matrix},$$

and similarly the blocks B_{21}, B_{22} and B_{23} yield the 2×12 cylindrical block

$$T_2 = \begin{matrix} (10) & (22) & (20) & (11) & (21) & (00) & (01) & (22) & (02) & (11) & (12) & (00) \\ (21) & (02) & (12) & (01) & (02) & (10) & (20) & (12) & (10) & (21) & (01) & (20) \end{matrix}.$$

The two cosets of $F_{(1)}$ are $\{(11),(20),(02)\}$, and $\{(10),(22),(01)\}$, and the two cosets of $F_{(2)}$ are $\{(10),(21),(02)\}$, and $\{(12),(20),(01)\}$.

We now choose $h_2^{(1)} = (11)$, and $h_3^{(1)} = (10)$ in the two cosets of $F_{(1)}$ and use T_1 to obtain the following two 2×12 cylindrical blocks:

$$T_{12} = \begin{matrix} (12) & (02) & (10) & (20) & (21) & (11) & (22) & (02) & (00) & (20) & (01) & (11) \\ (21) & (00) & (01) & (22) & (00) & (12) & (10) & (01) & (12) & (21) & (22) & (10) \end{matrix},$$

$$T_{13} = \begin{matrix} (11) & (01) & (12) & (22) & (20) & (10) & (21) & (01) & (02) & (22) & (00) & (10) \\ (20) & (02) & (00) & (21) & (02) & (11) & (12) & (00) & (11) & (20) & (21) & (12) \end{matrix}.$$

Similarly $h_2^{(2)} = (10)$ and $h_3^{(2)} = (01)$ in the two cosets of $F_{(2)}$ in conjunction with T_2 yield the following two 2×12 cylindrical blocks:

$$T_{22} = \begin{matrix} (20) & (02) & (00) & (21) & (01) & (10) & (11) & (02) & (12) & (21) & (22) & (10) \\ (01) & (12) & (22) & (11) & (12) & (20) & (00) & (22) & (20) & (01) & (11) & (00) \end{matrix},$$

$$T_{23} = \begin{matrix} (11) & (20) & (21) & (12) & (22) & (01) & (02) & (20) & (00) & (12) & (10) & (01) \\ (22) & (00) & (10) & (02) & (00) & (11) & (21) & (10) & (11) & (22) & (02) & (21) \end{matrix}.$$

The six 2×12 cylindrical blocks $T_{11} = T_1, T_{12}, T_{13}, T_{21} = T_2, T_{22}$, and T_{23} together are balanced for row, column, and diagonal neighbors (condition (b)) with no like neighbors (condition (a)) and hence give an $NBCD$.

As applications of Theorem 3.1, we have

Corollary 3.1. *Let $v = s^n \geq 5$ where s is an odd prime number and $n \geq 1$. Then, for every positive integer $m_1 \geq 2$, there exists an $NBCD$ for $v = s^n$ treatments in $b = (v-1)s^{n-1}/2$ cylindrical blocks of size $m_1 \times s$ each. When $n = 1$, the design is universally optimal in $D_1(v = s, b = (v-1)/2, m_1, m_2 = v)$.*

Proof: It is sufficient to give the $m_1 \times 2$ planar arrays $A_1, A_2, \ldots, A_{(v-1)/2}$ on F_v that satisfy all four conditions of Theorem 3.1. Choose $f_1 \in F_v - \{0, f, -f\}$ where f is any nonzero element of F_v. Now define $A_i = x^{i-1}A_1, i = 1, 2, \ldots, (v-1)/2$ where

$$A_1 = \begin{matrix} 0 & f_1 \\ f & f + f_1 \\ 2f & 2f + f_1 \\ \vdots & \vdots \\ (m_1 - 1)f & (m_1 - 1)f + f_1 \end{matrix}.$$

The proof is completed by checking the differences described in Theorem 3.1. □

Corollary 3.2. *Let $v = s^n = qm+1 \geq 5$ where s is an odd prime number, $q \geq 2$ and $n \geq 1$. Then there exists an $NBCD$ for $v = s^n$ treatments in $b = ms^{n-1}$ cylindrical blocks each of size $2 \times sq$. If $n = 1$, the design is universally optimal in $D_1(v, b = m, m_1 = 2, m_2 = vq)$.*

Proof: It is sufficient to give the m planar arrays $A_1, A_2, \ldots, A_m$ on F_v that satisfy all four conditions of Theorem 3.1. If $m = 1$, choose any $k > 1$ such that $x^k \neq -1$. If $m > 1$, choose any $k \not\equiv 0 \pmod{m}$ such that $x^k \neq -1$. In both cases take $f_1 = (1 - x^m)(1 + x^k)$ and define a $2 \times (q+1)$ planar array A as

$$A = \begin{matrix} x^m & x^{2m} & \ldots, & x^{qm} & x^m + f_1 \\ x^{m+k} & x^{2m+k} & \ldots, & x^{qm+k} & x^{m+k} + f_1 \end{matrix}.$$

Then the m arrays $A_i = x^{i-1}A$, $i = 1, 2, \ldots, m$ satisfy all conditions of Theorem 3.1. □

Corollary 3.3. *Let $v = s^n = 2qm + 1$ where s is a prime number, q is odd and $n \geq 1$. Then there exists an $NBCD$ for $v = s^n$ treatments in $b = ms^{n-1}$ cylindrical blocks each of size $2 \times sq$. If $n = 1$, the design is universally optimal in $D_1(v, b = m, m_1 = 2, m_2 = vq)$.*

Proof: Take $f_1 = (1 - x^{2m})(1 + x^k)$ where $k \not\equiv 0 \pmod{2m}$ satisfy $x^k \neq -1$. Then the m planar $2 \times (q+1)$ arrays $A_i = x^{i-1}A$, $i = 1, 2, \ldots, m$ satisfy all conditions of Theorem 3.1 where

$$A = \begin{matrix} x^{2m} & x^{4m} & \ldots, x^{2qm} & x^{2m} + f_1 \\ x^{2m+k} & x^{4m+k} & \ldots, x^{2qm+k} & x^{2m+k} + f_1 \end{matrix}.$$

□

We like to note that examples 1, 3, 4 and 5 are constructed using corollaries 3.1, 3.1, 3.3, and 3.2, respectively. Each cylindrical block of corollaries 3.2 and 3.3 has only two rows. The number of rows can be increased to any even number by taking g copies of each block and adjoining them vertically to construct a new cylindrical block with $2g$ rows. For the resulting design in $2g \times vq$ cylindrical blocks, the treatment replications and row-neighbor counts are g times those of corollaries 3.2-3, and the neighbor counts in columns and diagonals are $2g - 1$ times those of corollaries 3.2-3.

Next, we give two series constructions of $NBCD$s and universally optimal designs of Theorem 2.2. For the purpose of these constructions, we write $v = s^n + 1$ where s is an odd prime number and denote the v treatments by the v elements of $F_{s^n} \cup \{\infty\}$. Also, we use the convention that $\infty + f = \infty$ for all $f \in F_{s^n}$.

Theorem 3.2. *Let a prime number s and an integer $n \geq 1$ be such that $s^n \equiv 3 \pmod{4}$. Then there exists an $NBCD$ for $v = s^n + 1$ treatments in s^{n-1} $2 \times sv/2$ blocks. If $n = 1$, the design is universally optimal in $D_2(v = s + 1, b = 1, m_1 = 2, m_2 = v(v-1)/2)$.*

Proof: Define the following $2 \times (v/2+1)$ planar array

$$A = \begin{matrix} x^{v-4} & x^{v-2} & x^2 & \ldots, & x^{v-4} & x^{v-2} \\ \infty & -x^{v-2} & -x^2 & \ldots, & -x^{v-4} & \infty \end{matrix}.$$

Note that $\pm\{x^{v-2}, x^2, \ldots, x^{v-4}\} = \{x^{v-2}, x^1, x^2, \ldots, x^{v-3}\}$ since $s^3 \equiv 3 \bmod (4)$. Utilizing this result, we see that the neighbor differences among the finite elements of A are as follows:

the symmetric row neighbor differences are two copies of $F_{s^n} - \{0\}$,

the symmetric diagonal neighbor differences are two copies of $F_{s^n} - \{0\}$, and

the symmetric column neighbor differences are one copy of $F_{s^n} - \{0\}$.

Furthermore, the symbol ∞ is neighbored by two finite elements in rows as well as in diagonals and by one finite element in the first $v/2$ columns. Then the s planar arrays

$$B_j = A + (j-1)(1 - x^{v-4})J_{2\times(v/2+1)}, j = 1, 2, \ldots, s$$

can be merged, in the fashion of Theorem 3.1, at their common end columns to obtain a $2 \times sv/2$ cylindrical block. When $n = 1$, this block gives the desired universally optimal design. For $n \geq 2$, construct s^{n-1} blocks of $NBCD$ in the fashion of Theorem 3.1. □

Example 6. For $v = 8$, a 2×28 universally optimal design of Theorem 3.2 is (the symbol ∞ is replaced by 8 for convenience)

4 1 2 4 1 5 6 1 5 2 3 5 2 6 0 2 6 3 4 6 3 0 1 3 0 4 5 0
8 6 5 3 8 3 2 0 8 0 6 4 8 4 3 1 8 1 0 5 8 5 4 2 8 2 1 6

If $s^n \equiv 1 \pmod 4$, $s^n > 5$, then the finite row-neighbor differences obtained from A in Theorem 3.2 are all quadratic residues or all quadratic non-residues of F_{s^n}, and so are the column and diagonal neighbor differences. In this case, one may start with A and xA, and construct $2s^{n-1}$ cylindrical blocks using the technique described in the proof of Theorem 3.2. For $v = 5$, a universally optimal design in D_2 is given by example 2. Hence we state

Theorem 3.3. *Let s be a prime number and $n \geq 1$ be an integer such that $s^n \equiv 1$ (mod 4). Then there exists an $NBCD$ for $v = s^n + 1$ treatments in $2s^{n-1}$ $2 \times sv/2$ blocks. If $n = 1$, the design is universally optimal in $D_2(v = s+1, b = 2, m_1 = 2, m_2 = v(v-1)/2)$.*

In our next construction, we take optimal designs of Theorems 3.2 and 3.3 to obtain new $NBCD$s and universally optimal designs of Theorem 2.3.

Theorem 3.4. *An NBCD for $v = s^n + 1$, $b = s^{n-1}$, $m_1 = 2g$, $m_2 = sv/2$ can be constructed for every $g \geq 2$ whenever $s^n = 5$, or $s^n \equiv 3 \pmod 4$, and for the parameters $v = s^n + 1$, $b = 2s^{n-1}$, $m_1 = 2g$, $m_2 = vs/2$ for every $g \geq 2$ whenever $s^n \equiv 1 \pmod 4$, $s^n > 5$. If $n = 1$, these designs are universally optimal in D_3 having the respective parameters.*

Proof: For $s^n \equiv 3 \pmod 4$, take g copies of each block of the Theorem 3.2 design and adjoin them vertically to construct $2g \times vs/2$ cylindrical blocks. Similarly use g copies of the Theorem 3.3 design when $s^n \equiv 1 \pmod 4$, $s^n > 5$. For $s^n = 5$, use g copies of the Example 2 design. □

Finally, we give a series construction for $NBCD$s for $v = s^n + 1$ treatments where s is any odd prime number.

Theorem 3.5. *An NBCD for the parameters $v = s^n + 1$, $b = vs^{n-1}/2$, $m_1 = 2$, $m_2 = 2s$ can be constructed for every odd prime number s.*

Proof: For some $f_1 \in F_{s^n} - \{0, 1, s-1, s-3\}$, define $(s^n + 1)/2$ planar arrays on F_{s^n} as follows:

$$A_1 = \begin{matrix} \infty & 0 & \infty \\ 1 & 2 & s - f_1 - 1 \end{matrix}, \quad A_2 = \begin{matrix} \infty & 0 & \infty \\ s-1 & 2 & f_1 + 1 \end{matrix},$$

$$A_{i+1} = \begin{matrix} 0 & x^{i-1} & x^{i-1} f_1 \\ 2x^{i-1} & -x^{i-1} & x^{i-1}(2 + f_1) \end{matrix}, \quad i = 2, 3, \ldots, (s^n - 1)/2.$$

Using each planar array, construct s^{n-1} cylindrical blocks in the fashion of Theorems 3.1 and 3.2. That the $vs^{n-1}/2$ cylindrical blocks thus obtained from the above $v/2$ planar arrays is an $NBCD$ follows from the neighbor differences of the planar arrays. □

4. Efficiency Calculations.

In this section, we investigate numerically the behavior of some cylinder designs in the plane. An $m_1 \times m_2$ cylindrical block becomes an $m_1 \times m_2$ planar block when the cylinder is cut between any two columns. Write $\Delta(v, b, m_1, m_2)$ to denote the class of all connected designs for v treatments in b $m_1 \times m_2$ planar blocks. The planar versions of some cylinder designs constructed in section 3 will be compared with the corresponding hypothetical universally optimal design in $\Delta(v, b, m_1, m_2)$. We use the model (1.1) but with the planar error process given by $\sigma^2\Sigma^{-1} = I_b \otimes \Sigma_1^{-1}$ where

$$\Sigma_1^{-1} = I_{m_1 m_2} - \alpha_1(I_{m_1} \otimes H_{m_2}) - \alpha_2(H_{m_1} \otimes I_{m_2}) - \alpha_3(H_{m_1} \otimes H_{m_2}). \qquad (3.1)$$

Here $\alpha_1 > 0, \alpha_2 > 0, \alpha_3 > 0$, such that Σ_1^{-1} is positive definite, and H_t of order t satisfies

$$(H_t)_{ll'} = \begin{cases} 1, & \text{if } (l - l') = \pm 1, \\ 0, & \text{otherwise,} \end{cases} .$$

This three parameter autonormal error process has been used previously by Morgan and Uddin (1998), Uddin and Morgan (1997b), by Gill and Shukla (1985) under the model (1.1) with $\beta = 0$, and by Uddin and Morgan (1991) with $\alpha_1 = \alpha_2$ in their investigation of optimal planar designs. The A-, E-, and D-efficiencies of the planar versions of cylinder designs of examples 1- 5 are calculated here under this error process.

The common nonzero eigenroot θ of the C-matrix of a hypothetical universally optimal design in $\Delta(v, b, m_1, m_2)$ arrived at by the method of Kiefer (1975) would have the following upper bound (see Morgan and Uddin, 1991, pp 2176):

$$\theta \leq \frac{b}{v-1}[\text{trace}(\Sigma_1^{-1}) - \frac{J_{1\times m_1 m_2}\Sigma_1^{-1}J_{m_1 m_2 \times 1}}{v}] = \theta^* \text{ (say)}.$$

The nonzero eigenroots $\theta_1, \theta_2, \ldots, \theta_{v-1}$ of the C-matrix of a proposed design can be compared with θ^* to obtain lower bounds for A-, E- and D-efficiencies (see Morgan and Uddin, 1991) as follows:

$$A_{\text{efficiency}} \geq \frac{v-1}{\sum_{i=1}^{v-1}(\theta^*/\theta_i)}, \quad E_{\text{efficiency}} \geq \frac{\min(\theta_1,\ldots,\theta_{v-1})}{\theta^*}, \quad D_{\text{efficiency}} \geq [\prod_{i=1}^{v-1}(\frac{\theta_i}{\theta^*})]^{\frac{1}{v-1}}.$$

These lower bound efficiencies of the planar versions (as displayed) of cylinder designs of examples 1- 5 are calculated for some combinations of α_1, α_2 and α_3 for which Σ_1 is positive definite. These results are reported in Table 1 below where each entry multiplied by 1000 is in the order (A, E, D) and truncated after three decimal places. All these designs perform very well in planar applications especially with respect to A and D–efficiencies. The poor performance (especially with respect to E) of example 3 is due to the fact that the rows of this design are very short, so that making the cylinder blocks planar has a heavier cost than for the other examples. Furthermore, this design was optimal within the subclass of equireplicate-block (cylindrical) designs only.

Table 1 : Efficiencies of planar versions of cylinder designs of examples 1 - 5.

	planar versions of designs of examples				
α_1 α_2 α_3	1	2	3	4	5
.1 .1 .05	(999, 989, 999)	(999, 975, 999)	(997, 977, 997)	(999, 978, 999)	(997, 979, 997)
.1 .2 .05	(999, 989, 999)	(999, 976, 999)	(994, 973, 994)	(999, 979, 999)	(997, 980, 997)
.1 .3 .05	(999, 989, 999)	(999, 976, 999)	(988, 965, 988)	(999, 979, 999)	(998, 981, 998)
.1 .4 .05	(999, 989, 999)	(999, 977, 999)	(973, 946, 974)	(999, 979, 999)	(998, 982, 998)
.2 .1 .05	(999, 970, 999)	(999, 961, 999)	(993, 955, 994)	(999, 965, 999)	(997, 967, 997)
.2 .2 .05	(998, 971, 999)	(999, 961, 999)	(989, 948, 989)	(999, 966, 999)	(997, 968, 997)
.2 .2 .10	(998, 978, 998)	(998, 954, 999)	(978, 932, 979)	(999, 959, 999)	(997, 967, 997)
.2 .2 .15	(995, 975, 995)	(998, 948, 998)	(957, 898, 958)	(998, 953, 998)	(996, 965, 996)
.2 .3 .05	(998, 971, 998)	(998, 962, 999)	(977, 930, 977)	(999, 966, 999)	(997, 969, 998)
.3 .1 .05	(997, 950, 997)	(997, 947, 998)	(986, 929, 987)	(998, 953, 999)	(996, 955, 996)
.3 .2 .05	(996, 949, 996)	(997, 948, 998)	(975, 913, 977)	(998, 953, 998)	(995, 956, 996)
.4 .1 .05	(991, 926, 993)	(992, 934, 993)	(970, 893, 972)	(993, 941, 994)	(988, 940, 989)

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